

Marchenko–Pastur Law for Spectra of Random Weighted Bipartite Graphs

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Abstract—We study the spectra of random weighted bipartite graphs. We establish that, under specific assumptions on the edge probabilities, the symmetrized empirical spectral distribution function of the graph’s adjacency matrix converges to the symmetrized Marchenko–Pastur distribution function.

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1. INTRODUCTION

In traditional graph theory, we assume graphs to be static and deterministic. However, most real-world networks exhibit some degree of randomness. Social networks can have random connections based on chance encounters. The internet has random connections based on routing protocols. Focusing solely on deterministic graphs wouldn’t capture the complexity of these systems. Random graph models allow for development of algorithms that are more efficient in handling massive datasets with millions or billions of vertices. In summary, while deterministic graphs are valuable for foundational studies, random graphs become essential for capturing the inherent randomness and large scale of real-world networks.

Over the past three decades, there has been a increasing number of literature dedicated to exploring random graphs. Random graph theory has experienced rapid and substantial development in recent years, driven by its diverse applications across various fields ranging from mathematics and physics to finance, social sciences, biology, chemistry, information networks (see e.g. [5, 6]). The foundation of random graph theory dates back to the 1950s, established by Erdős and Rényi [2], blending principles of graph theory and probability theory into an interdisciplinary domain. The Erdős–Rényi random graph is a graph with n vertices where edges are independently chosen with a probability p , lying in the range $0 < p < 1$. This edge probability can either be constant or a function dependent on n . Spectral graph theory emerges as a pivotal area of study within graph theory, focusing on the graph’s properties concerning the eigenvalues and eigenvectors of associated matrices. Various matrices, such as the adjacency matrix, Laplacian matrix, and normalized Laplacian matrix, can be linked to a given graph. The spectra of these matrices are called the spectra of the graph. Understanding the spectra of random graphs is crucial to understanding the properties of random graphs. For instance, the spectrum of the adjacency and Laplacian matrices of a graph correlates with its connectivity and the number of occurrences of specific subgraphs, and also with its chromatic number and its independence number. The spectrum of the normalized Laplacian matrix is related to diffusion on graphs, random walks on graphs [6].

This paper explores random weighted bipartite graphs. These graphs consist of two separate sets with n and m vertices each with no connections within the same set, resulting in a total of $n + m$

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vertices in the bipartite graph. The connections between vertices occur randomly, and the adjacency matrices of these graphs have a block structure. This vertices segregation into two parts is valuable as it mirrors a prevalent characteristic observed in numerous real-world networks, from social interactions where users connect with other users, to recommendation systems matching products with interested customers. Bipartite graphs offer a powerful lens to model these inherent relationships [1, 7]. Adding weights to a bipartite graph significantly enhances its utility, enabling the quantification of relationship strength, prioritization of specific interactions, and other suitable applications.

In [8] the convergence of the empirical spectral distribution function of the generalized random graph's adjacency matrix to the distribution function of the semicircular Wigner law was proved under a number of conditions on probabilities of the graph edges. The present work investigates random weighted bipartite graphs, extending the result established in [8] on the convergence of the symmetrized empirical spectral distribution function of the bipartite graph's adjacency matrix to the symmetrized Marchenko–Pastur distribution function.

Consider an undirected, weighted bipartite graph $G = (V_1, V_2, E, \omega)$, where V_1 and V_2 are finite vertex sets with $|V_1| = n$, $|V_2| = m$, and n, m are quantities of the same order. Without loss of generality, assume $m \geq n$. We denote the graph's adjacency matrix by $\mathbf{A}$, which exhibits the following block structure:

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_n & \mathbf{W} \\ \mathbf{W}^* & \mathbf{O}_m \end{bmatrix}.$$

Here, $\mathbf{O}_n$ and $\mathbf{O}_m$ represent zero matrices of sizes $n \times n$ and $m \times m$, respectively. $\mathbf{W}$ is an $n \times m$ matrix defined as $\mathbf{W} = \frac{1}{\sqrt{a_n}} \Xi \circ \mathbf{X}$, where $\mathbf{X} = (X_{jk})$ be a matrix of size $n \times m$ with independent elements X_{jk} . We assume that $\mathbb{E}X_{jk} = 0$ and $\mathbb{E}X_{jk}^2 = 1$ for all $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$. Additionally, let $\Xi = (\xi_{jk})$ be another $n \times m$ matrix with independent Bernoulli entries ξ_{jk} , that is ξ_{jk} take the value 1 with probabilities $\mathbb{E}\xi_{jk} = p_{jk}$ and the value 0 with probabilities $1 - p_{jk}$. We define a positive scalar value a_n as

$$a_n = \frac{1}{n} \sum_{j=1}^n \sum_{k=1}^m p_{jk},$$

and assume $m = m(n)$ with $\lim_{n \rightarrow \infty} n/m =: y$, where y is a constant.

Let $s_1 \geq s_2 \geq \dots \geq s_n$ denote the singular values of matrix $\mathbf{W}$, ordered in non-increasing order. We introduce the corresponding symmetrized empirical spectral distribution (ESD) function:

$$F_n(x) := \frac{1}{2n} \sum_{k=1}^n \left(\mathbf{1}\{s_k \leq x\} + \mathbf{1}\{-s_k \leq x\} \right).$$

Here, $\mathbf{1}\{\cdot\}$ denotes the indicator function. Furthermore, we define $G_y(x)$ as the symmetrized Marchenko–Pastur (MP) distribution function with density:

$$g_y(x) = \frac{1}{2\pi y|x|} \sqrt{(x^2 - a^2)(b^2 - x^2)} \mathbf{1}\{a^2 \leq x^2 \leq b^2\},$$

where $a = 1 - \sqrt{y}$ and $b = 1 + \sqrt{y}$.

We establish the following conditions that will be used throughout the paper:

- Condition $CP(0)$:

$$a_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

- Condition $CP(1)$:

$$\lim_{n \rightarrow \infty} \frac{1}{na_n} \sum_{j=1}^n \sum_{k=1}^m \left| p_{jk} - \frac{a_n}{m} \right| = 0.$$

- Condition $CX(0)$: for any $\tau > 0$

$$L_n(\tau) := \frac{1}{na_n} \sum_{j=1}^n \sum_{k=1}^m p_{jk} \mathbb{E} X_{jk}^2 \mathbf{1}\{|X_{jk}| > \tau \sqrt{a_n}\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The main result of the paper is as follows.

Theorem. *Under conditions $CP(0)$, $CP(1)$, and $CX(0)$, the following limit relation holds:*

$$\sup_x |F_n(x) - G_y(x)| \rightarrow 0 \text{ in probability as } n \rightarrow \infty.$$

2. PROOF OF THE THEOREM

Let $\mathbf{R}_{\mathbf{A}}(z) = (\mathbf{A} - z\mathbf{I}_{n+m})^{-1}$ be the resolvent matrix of $\mathbf{A}$. The Schur complement gives

$$\mathbf{R}_{\mathbf{A}}(z) = \begin{bmatrix} z(\mathbf{W}\mathbf{W}^* - z^2\mathbf{I}_n)^{-1} & \mathbf{W}(\mathbf{W}^*\mathbf{W} - z^2\mathbf{I}_m)^{-1} \\ (\mathbf{W}^*\mathbf{W} - z^2\mathbf{I}_m)^{-1}\mathbf{W}^* & z(\mathbf{W}^*\mathbf{W} - z^2\mathbf{I}_m)^{-1} \end{bmatrix}. \quad (1)$$

Denote by $s_y(z)$ the Stieltjes transform of the symmetrized Marchenko–Pastur distribution function $G_y(x)$ and by $s_n(z)$ the Stieltjes transform of the distribution function $F_n(x)$. The Stieltjes transform of $G_y(x)$ is given by

$$s_y(z) = \frac{-z + \frac{1-y}{z} + \sqrt{(z - \frac{1-y}{z})^2 - 4y}}{2y},$$

and the Stieltjes transform of $F_n(x)$ — by

$$s_n(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF_n(x).$$

Let $j \in \{1, \dots, n\}$. Consider σ -algebra $\mathfrak{M}^{(j)}$, generated by the elements of $\mathbf{A}$ with the exception of the row and the column with number j . By symbol $\mathbf{A}^{(j)}$ we denote the matrix $\mathbf{A}$ which row and column with number j are deleted. In a similar way, we will denote all objects defined via $\mathbf{A}^{(j)}$, such that the resolvent matrix $\mathbf{R}^{(j)}$, the ESD Stieltjes transform $s_n^{(j)}$ and so on. The symbol $\mathbb{E}_j$ denotes the conditional expectation with respect to the σ -algebra $\mathfrak{M}_j$.

By equation (1), we have:

$$s_n(z) = \frac{1}{n} \sum_{j=1}^n R_{jj} = \frac{1}{n} \sum_{l=1}^m R_{l+n, l+n} + \frac{m-n}{nz},$$

and

$$\begin{aligned} R_{jj}(z) &= \frac{1}{-z - \frac{1}{a_n} \sum_{l,k=1}^m X_{jk} X_{jl} \xi_{jk} \xi_{jl} R_{k+n, l+n}^{(j)}} \\ &= -\frac{1}{z + y s_n(z) - \frac{1-y}{z}} (1 - \varepsilon_j R_{jj}(z)), \end{aligned}$$

where

$$\varepsilon_j = \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4} + \varepsilon_{j5},$$

$$\varepsilon_{j1} = -\frac{1}{a_n} \sum_{k=1}^m X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n, k+n}^{(j)},$$

$$\begin{aligned}\varepsilon_{j2} &= -\frac{1}{a_n} \sum_{k=1}^m p_{jk} (X_{jk}^2 - 1) R_{k+n, k+n}^{(j)}, \\ \varepsilon_{j3} &= -\frac{1}{a_n} \sum_{k=1}^m \left(p_{jk} - \frac{a_n}{m} \right) R_{k+n, k+n}^{(j)}, \\ \varepsilon_{j4} &= \frac{1}{m} \sum_{k=1}^m R_{k+n, k+n} - \frac{1}{m} \sum_{k=1}^m R_{k+n, k+n}^{(j)}, \\ \varepsilon_{j5} &= -\frac{1}{a_n} \sum_{l \neq k}^m X_{jk} X_{jl} \xi_{jk} \xi_{jl} R_{k+n, l+n}^{(j)}.\end{aligned}$$

Note that the following inequality holds (see [4], Lemma 7.6):

$$\max_{k,l} \left\{ |R_{k+n, l+n}(z)|, |R_{k+n, l+n}^{(j)}(z)| \right\} \leq \frac{1}{v}, \quad z = u + iv \quad v > 0. \quad (2)$$

Upon summation of $R_{jj}(z)$, we obtain

$$s_n(z) = -\frac{1}{z + y s_n(z) - \frac{1-y}{z}} (1 - T_n(z)),$$

where

$$T_n(z) = \frac{1}{n} \sum_{j=1}^n \varepsilon_j R_{jj} = \sum_{\nu=1}^5 T_{n\nu}(z), \quad T_{n\nu}(z) = \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}.$$

Since the Stieltjes transform $s_y(z)$ of the symmetrized Marchenko–Pastur distribution satisfies the equation

$$s_y(z) = -\frac{1}{z + y s_y(z) - \frac{1-y}{z}},$$

showing that $\mathbb{E}|T_n(z)| \rightarrow 0$ uniformly in $z = u + iv$, $|u| \leq u_0$, $v \geq C$, is sufficient to prove the convergence of $s_n(z) \rightarrow s_y(z)$. To do this, we need a number of auxiliary assertions.

Lemma 1. *Let the conditions $CP(0)$, $CP(1)$, and $CX(0)$ be satisfied. Then*

$$\mathbb{E}|T_{n1}| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. One has

$$\begin{aligned}\mathbb{E}|T_{n1}| &\leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}|\varepsilon_{j1} R_{jj}| \leq \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n, k+n}^{(j)} \right| \\ &\leq \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n, k+n}^{(j)} \mathbf{1}\{|X_{jk}| \leq \tau \sqrt{a_n}\} \right| \\ &\quad + \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n, k+n}^{(j)} \mathbf{1}\{|X_{jk}| > \tau \sqrt{a_n}\} \right| \\ &\leq \frac{1}{va_n} \left(\frac{1}{n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n, k+n}^{(j)} \mathbf{1}\{|X_{jk}| \leq \tau \sqrt{a_n}\} \right|^2 \right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{nva_n} \sum_{j=1}^n \sum_{k=1}^m \mathbb{E} \left| X_{jk}^2 (\xi_{jk} - p_{jk}) R_{k+n,k+n}^{(j)} \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\} \right| \\
& \leq \frac{1}{v^2 a_n} \left(\frac{1}{n} \sum_{j=1}^n \sum_{k=1}^m \mathbb{E} (\xi_{jk} - p_{jk})^2 \mathbb{E} X_{jk}^4 \mathbf{1}\{|X_{jk}| \leq \tau\sqrt{a_n}\} \right)^{\frac{1}{2}} \\
& + \frac{1}{nv^2 a_n} \sum_{j=1}^n \sum_{k=1}^m \mathbb{E} |\xi_{jk} - p_{jk}| \mathbb{E} X_{jk}^2 \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\} \\
& \leq \frac{\tau\sqrt{a_n}}{v^2 a_n} \left(\frac{1}{n} \sum_{j=1}^n \sum_{k=1}^m p_{jk}(1 - p_{jk}) \right)^{\frac{1}{2}} \\
& + \frac{2}{nv^2 a_n} \sum_{j=1}^n \sum_{k=1}^m p_{jk}(1 - p_{jk}) \mathbb{E} X_{jk}^2 \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\} \\
& \leq \frac{\tau + 2L_n(\tau)}{v^2}.
\end{aligned}$$

Note that we may substitute τ in condition $CX(0)$ by a decreasing sequence τ_n tending to zero such that

$$L_n(\tau_n) \rightarrow 0$$

as $n \rightarrow \infty$. This concludes the proof of the lemma. $\square$

Lemma 2. *Let the conditions $CP(0)$, $CP(1)$, and $CX(0)$ be satisfied. Then*

$$\mathbb{E}|T_{n2}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. We define random variables $\eta_{jk} = (X_{jk}^2 - 1) \mathbf{1}\{|X_{jk}| \leq \tau\sqrt{a_n}\}$. It is easy to see that $\mathbb{E}\eta_{jk} = -\mathbb{E}(X_{jk}^2 - 1) \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\}$ and $|\eta_{jk} - \mathbb{E}\eta_{jk}| \leq 2\tau^2 a_n$. From (2) and these relations we obtain

$$\begin{aligned}
\mathbb{E}|T_{n2}| & \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j2} R_{jj}| \leq \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m p_{jk} (X_{jk}^2 - 1) R_{k+n,k+n}^{(j)} \right| \\
& \leq \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m p_{jk} (X_{jk}^2 - 1) R_{k+n,k+n}^{(j)} \mathbf{1}\{|X_{jk}| \leq \tau\sqrt{a_n}\} \right| \\
& + \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m p_{jk} (X_{jk}^2 - 1) R_{k+n,k+n}^{(j)} \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\} \right| \\
& \leq \frac{1}{nva_n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} \left| \sum_{k=1}^m p_{jk} R_{k+n,k+n}^{(j)} (\eta_{jk} - \mathbb{E}\eta_{jk}) \right|^2 \\
& + \frac{2}{nva_n} \sum_{j=1}^n \sum_{k=1}^m p_{jk} \mathbb{E} |X_{jk}^2 - 1| \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\} \mathbb{E} |R_{k+n,k+n}^{(j)}|.
\end{aligned}$$

Applying Cauchy's inequality and (2), we get

$$\begin{aligned}
\mathbb{E}|T_{n2}| & \leq \frac{1}{va_n} \left(\frac{1}{n} \sum_{j=1}^n \sum_{k=1}^m p_{jk}^2 \mathbb{E} |R_{k+n,k+n}^{(j)}|^2 \mathbb{E} |\eta_{jk} - \mathbb{E}\eta_{jk}|^2 \right)^{\frac{1}{2}} \\
& + \frac{2}{nv^2 a_n} \sum_{j=1}^n \sum_{k=1}^m p_{jk} \mathbb{E} X_{jk}^2 \mathbf{1}\{|X_{jk}| > \tau\sqrt{a_n}\}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{2}{nv^2 a_n} \sum_{j=1}^n \sum_{k=1}^m p_{jk} \mathbf{P}\{|X_{jk}| > \tau \sqrt{a_n}\} \\
 & \leq \frac{C}{v^2} \left(\tau + \frac{L_n(\tau)}{\tau^2 a_n} + L_n(\tau) \right).
 \end{aligned}$$

Note that we can substitute τ in condition $CX(0)$ by a decreasing sequence τ_n tending to zero such that

$$\tau_n^2 a_n \rightarrow \infty$$

and

$$L_n(\tau_n) \rightarrow 0$$

as $n \rightarrow \infty$. This concludes the proof of the lemma. $\square$

Lemma 3. *Let the conditions $CP(0)$, $CP(1)$, and $CX(0)$ be satisfied. Then*

$$\mathbb{E}|T_{n3}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. By the definition of T_{n3} and inequality (2), we have

$$\begin{aligned}
 \mathbb{E}|T_{n3}| & \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}|\varepsilon_{j3} R_{jj}| \leq \frac{1}{nv a_n} \sum_{j=1}^n \sum_{k=1}^m \left| p_{jk} - \frac{a_n}{m} \right| \mathbb{E}|R_{k+n, k+n}^{(j)}| \\
 & \leq \frac{1}{nv^2 a_n} \sum_{j=1}^n \sum_{k=1}^m \left| p_{jk} - \frac{a_n}{m} \right|.
 \end{aligned}$$

Combining this inequality and condition $CP(1)$, we obtain the required result. $\square$

Lemma 4. *Let the conditions $CP(0)$, $CP(1)$, and $CX(0)$ be satisfied. Then*

$$\mathbb{E}|T_{n4}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Using Lemma 3.2 from paper [3] to estimate the resolvents traces difference, we obtain

$$\mathbb{E}|T_{n4}| \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}|\varepsilon_{j4} R_{jj}| \leq \frac{1}{nmv} \sum_{j=1}^n \mathbb{E} \left| \sum_{k=1}^m R_{k+n, k+n} - \sum_{k=1}^m R_{k+n, k+n}^{(j)} \right| \leq \frac{y}{nv^2}.$$

This concludes the proof of the Lemma. $\square$

Lemma 5. *Let the conditions $CP(0)$, $CP(1)$, and $CX(0)$ be satisfied. Then*

$$\mathbb{E}|T_{n5}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Applying the triangle inequality, Cauchy's inequality, inequality (2) and computing the variance of the quadratic form with respect to the variables $\{X_{jk} \xi_{jk}\}_{k=1}^m$ conditioned on fixed values of the matrices $\mathbf{R}^{(j)}$, we obtain

$$\begin{aligned}
 \mathbb{E}|T_{n5}| & \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}|\varepsilon_{j5} R_{jj}| \leq \frac{1}{a_n v} \left(\frac{1}{n} \sum_{j=1}^n \mathbb{E} \left| \sum_{l \neq k}^m X_{jk} X_{jl} \xi_{jk} \xi_{jl} R_{k+n, l+n}^{(j)} \right|^2 \right)^{\frac{1}{2}} \\
 & \leq \frac{1}{a_n v} \left(\frac{1}{n} \sum_{j=1}^n \sum_{l \neq k}^m p_{jk} p_{jl} \mathbb{E}|R_{k+n, l+n}^{(j)}|^2 \right)^{\frac{1}{2}} \leq \frac{1}{a_n^{1/2} v^2}.
 \end{aligned}$$

Here we use the inequality (see [4], Lemma 7.6)

$$\sum_l^m p_{jl} |R_{k+n,l+n}^{(j)}|^2 \leq \frac{\operatorname{Im} R_{k+n,k+n}^{(j)}}{v} \leq \frac{1}{v^2}.$$

This concludes the proof of the lemma. □

We are now ready to prove the main result.

Proof of Theorem. First, we note that the following two equations are valid:

$$ys_y^2(z) + \left(z - \frac{1-y}{z}\right)s_y(z) + 1 = 0,$$

$$ys_n^2(z) + \left(z - \frac{1-y}{z}\right)s_n(z) + 1 = T_n(z).$$

These equations yield

$$\begin{aligned} & \left| \left(z - \frac{1-y}{z}\right) - \frac{2y}{v} \right| |s_n(z) - s_y(z)| \\ & \leq \left| y(s_n^2(z) - s_y^2(z)) + \left(z - \frac{1-y}{z}\right)(s_n(z) - s_y(z)) \right| = |T_n(z)|. \end{aligned}$$

Choosing $v_0 \geq \sqrt{2(1-y)}$, we have

$$|s_n(z) - s_y(z)| \leq C|T_n(z)|. \quad (3)$$

Since $\mathbb{E}|T_n(z)| \leq \sum_{\nu=1}^5 \mathbb{E}|T_{n\nu}(z)|$, based on Lemmas 1–5 we have $\mathbb{E}|T_n(z)| \rightarrow 0$. From here and (3) it follows that

$$\mathbb{E}|s_n(z) - s_y(z)| \rightarrow 0 \text{ as } n \rightarrow \infty$$

uniformly in $z = u + iv$, $|u| \leq u_0$, and $v \geq v_0$. Since $s_n(z)$ are analytic functions and $|s_n(z)| \leq \frac{1}{v_0}$ for any $z = u + iv$ and $v \geq v_0$, it follows from Montel's theorem that the family of locally bounded analytic functions $\{s_n(z)\}$ is a precompact subset. This implies the convergence in probability

$$s_n(z) \rightarrow s_y(z) \text{ as } n \rightarrow \infty$$

for $z = u + iv$ and $v \geq v_0$. Now, convergence of Stieltjes transforms $s_n(z)$ implies convergence of the symmetrized empirical distribution functions $F_n(x)$ to the symmetrized Marchenko–Pastur distribution function in probability. The theorem is proved.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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