

Article

On the Limiting Distribution of the Spectra of Random Block Matrices

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Abstract

The behavior of the spectra of symmetric block-type random matrices with symmetric blocks of high dimensionality is considered in this paper. Under minimal conditions regarding the distributions of matrix block elements (Lindeberg conditions), the universality of the limiting empirical distribution function of block-type random matrices is shown.

Keywords: block matrices; Toeplitz matrices; Hankel matrices; spectral distributions

MSC: 60B20; 60F99

1. Introduction

In the theory of high-dimensional random matrices, one of the main objects to study is the empirical spectral distribution function of high-dimensional random matrices and its approximation. The most studied matrices are ensembles of symmetric real or Hermite matrices with independent elements (semicircular law) and sample covariance matrices (Marchenko–Pastur distribution law). We consider ensembles of random matrices of a rather wide class—matrices with a block structure. The study of the spectra of the block-type random matrices began at the end of the last century and the beginning of the 2000s. One of the first works in this field was Girko’s paper [1]. He considered matrices of increasing dimensions with block elements of fixed dimensions and proved various laws for the limiting spectral distribution of such matrices. In the first decade of the 21st century, a number of papers appeared focused on the study of the spectra of block matrices with blocks of increasing dimensions. See, for example, the works [2–6]. The study of such matrix models is gaining particular popularity in connection with various applications in machine learning and neural networks. See, for instance, [7,8].

One such example is communication channels with multiple transmitting and receiving antennas, the so-called MIMO (Multi-Input, Multi-Output) channels. The channel matrix has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_L & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_L & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_L & \dots & \mathbf{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{O} & \mathbf{O} & \dots & \dots & \mathbf{O} & \mathbf{A}_1 & \dots & \mathbf{A}_L \end{bmatrix},$$

where $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_L$ are matrices of size $n \times n$ and $\mathbf{A}_v = (h_{ij}^{(v)})_{i,j=1}^n$, where $h_{ij}^{(v)}$ reflects the channel’s effect on the signal transmitted from antenna j of the transmitter and received at antenna i of the receiver.



Academic Editor: Jin-Ting Zhang

Received: 31 March 2025

Revised: 14 May 2025

Accepted: 2 June 2025

Published: 20 June 2025

Citation: Tikhomirov, A.N. On the Limiting Distribution of the Spectra of Random Block Matrices. *Mathematics* **2025**, *13*, 2056. <https://doi.org/10.3390/math13132056>

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In what follows, we denote matrices and vectors in bold. To calculate the capacity of a channel, one needs to know the distribution of the eigenvalues of the matrix, as follows:

$$\mathbf{H} = \begin{bmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{O} \end{bmatrix},$$

where $\mathbf{O}$ denotes a matrix with zero entries and $\mathbf{A}^*$ denotes the complex conjugate of $\mathbf{A}$. For details, see, e.g., [4]. Some more examples of block matrix applications can be found in spectral graph theory. Adjacency matrices and Laplace matrices of random multipartite graphs are examples of block-type random matrices; see [9–11]. Block random matrix models are applicable to the study of products of random matrices due to so-called linearization. The simplest example is sample covariance matrices, where instead of matrices of the form $\mathbf{X}\mathbf{X}^*$, matrices of the form $\mathbf{H} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix}$ are studied; see [12] and the references therein for more details. The asymptotics of the spectra of the products of random matrices are studied, for example, when analyzing the asymptotics of the information capacity of communication channels; see e.g., [13,14]. Block matrices play a fundamental role in neural networks, particularly in large-scale deep learning, optimization, and structured architectures. Their applications span distributed training, sparse computation, attention mechanisms, and optimization theory, making them essential for modern machine learning systems. Recent work, such as that by Xiao-Kai An et al. [15], has explored nonconvex optimization techniques for analyzing spectral distributions in random matrices, which have implications for neural network training dynamics.

The development of methods for studying the spectra of random classical matrix ensembles (Wigner random matrices, sampled covariance matrices) in recent years has led to an increased interest in studying the spectra of random matrices with block structures. See, for example, [6,16–26] and references therein.

1.1. List of Notations

We denote matrices by capital letters in bold, (column) vectors by lowercase letters in bold, and matrix elements by the same letters as matrices (as a rule) but in regular font.

The symbol $\mathbf{I}$ denotes the unit matrix; when it is necessary to specify the dimensions, we provide a lower index; e.g., $\mathbf{I}_{n-1}$ means a unit matrix of dimensions $n - 1$.

With $\mathbf{R}_\mathbf{A}$, we denote the resolvent matrix for matrix $\mathbf{A}$, i.e., $\mathbf{R}_\mathbf{A} = (\mathbf{A} - z\mathbf{I})$.

With $\mathbf{e}_j$, $j = 1, \dots, k$, we denote the standard base vectors in $\mathbb{R}^k$ (j -th component equals one, all the others are zeros)

With $\mathbf{E}^{(ij)}$, we denote the matrix defined as $\mathbf{E}^{(ij)} = \mathbf{e}_i \mathbf{e}_j^T$, i.e., all the elements of $\mathbf{E}^{(ij)}$ are zeros, except for $E_{ij}^{(ij)} = 1$.

For any vector $\mathbf{a}$ and any matrix $\mathbf{A}$, we denote the transposed vector $\mathbf{a}$ and the transposed matrix $\mathbf{A}$ by $\mathbf{a}^T$ and $\mathbf{A}^T$, respectively. For a complex matrix $\mathbf{A}$, we denote the Hermitian conjugate of $\mathbf{A}$ by $\mathbf{A}^*$, i.e., $\mathbf{A}^* = (\overline{\mathbf{A}})^T$.

We denote the indicator function by $\mathbb{I}\{\cdot\}$.

1.2. Model Representation

Consider the following model of block random matrices. Let $\mathbf{X}^{(ij)}$, $i, j = 1, \dots, k$, be a family of $n \times n$ random matrices. We will assume that for any fixed pair of indices (i, j) , $i, j = 1, \dots, k$, the elements of matrix $\mathbf{X}^{(ij)}$ are independent (with respect to symmetry) and have a mean of zero and finite variance, i.e.,

$$\mathbb{E}X_{pq}^{(i,j)} = 0 \text{ and } \mathbb{E}(X_{pq}^{(i,j)})^2 = (\sigma_{pq}^{(i,j)})^2 < \infty. \quad (1)$$

We will also assume that the matrices $\mathbf{X}^{(i,j)}$ either coincide at $(i,j) \neq (i_1, j_1)$ or are independent.

Let the symbol $\otimes$ denote the Kronecker product. Consider a matrix of the form

$$\mathbf{W}_{\mathbf{X}} = \sum_{i,j=1}^k \mathbf{E}^{(i,j)} \otimes \mathbf{X}^{(i,j)}. \quad (2)$$

We find it more convenient to change the representation of (2). Since we have assumed that some of the matrices $\mathbf{X}^{(i,j)}$ may coincide, we define the so-called “generating matrices” $\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(m)}$ (with $m \leq k^2$), which are random and independent, such that for any pair of indices (i,j) : $i, j = 1, \dots, k$, there exists some $l = l(i,j)$: $1 \leq l \leq m$ with $\mathbf{X}^{(i,j)} = \mathbf{Z}^{(l)}$. Suppose, for $l = 1, \dots, m$,

$$\mathcal{A}_l = \{(i,j) : i, j = 1, \dots, k : \mathbf{X}^{(i,j)} = \mathbf{Z}^{(l)}\}. \quad (3)$$

Then the matrix $\mathbf{W}$ can be rewritten as

$$\mathbf{W} = \sum_{l=1}^m \left(\sum_{(i,j) \in \mathcal{A}_l} \mathbf{E}^{(i,j)} \right) \otimes \mathbf{Z}^{(l)}.$$

1.3. Toeplitz and Hankel Random Matrices

The Toeplitz and Hankel random matrices occupy a special place in the study of structured matrices. In [3], the existence of a limiting spectral distribution for random Hermitian Toeplitz and Hankel random matrices was proven. In particular, it was proven that the limiting distribution of the Toeplitz and Hankel matrices has infinite support and differs from the normal distribution (the variance is 1, but the fourth moment is 8/3). Oraby in [5] showed that for circulant block-type Toeplitz matrices with random blocks of high dimensionality, the limiting spectral distribution is a mixture of semicircular laws with variances depending on the dimensionality of the original Toeplitz matrix. In [27], Oraby’s result was obtained under weaker conditions for the distribution of block elements. In [28], it was shown that for a palindromic Toeplitz matrix with Wigner blocks, the limiting spectral distribution is normal. In [18,29], symmetric block Toeplitz and Hankel matrices were considered. Let the generating matrices be $\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(k)}$. Define the multiplicity sets as $\mathcal{A}_l = \{(i,j) : |i-j| = l+1\}$, for $l = 1, \dots, k$. The symmetric Toeplitz matrix has the form

$$\mathbf{W} = \sum_{l=1}^k \mathbf{Z}^{(l)} \otimes \left(\sum_{(i,j) \in \mathcal{A}_l} \mathbf{E}^{(i,j)} \right) = \begin{bmatrix} \mathbf{Z}^{(1)} & \mathbf{Z}^{(2)} & \dots & \mathbf{Z}^{(k-1)} & \mathbf{Z}^{(k)} \\ \mathbf{Z}^{(2)} & \mathbf{Z}^{(1)} & \dots & \mathbf{Z}^{(k-2)} & \mathbf{Z}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{Z}^{(k)} & \mathbf{Z}^{(k-1)} & \dots & \mathbf{Z}^{(2)} & \mathbf{Z}^{(1)} \end{bmatrix}. \quad (4)$$

Hankel block matrices and circulant block matrices can be represented in a similar way.

1.3.1. Hankel Matrices

In a Hankel matrix, the number of distinct blocks is $m = 2k - 1$, and there are equal blocks on all diagonals perpendicular to the main diagonal, $\mathcal{A}_l = \{(i, j) : i + j = l + 1\}$, $l = 1, \dots, 2k - 1$:

$$\mathbf{W}_X = \begin{bmatrix} \mathbf{Z}^{(1)} & \mathbf{Z}^{(2)} & \mathbf{Z}^{(3)} & \dots & \mathbf{Z}^{(k-1)} & \mathbf{Z}^{(k)} \\ \mathbf{Z}^{(2)} & \mathbf{Z}^{(3)} & \mathbf{Z}^{(4)} & \dots & \mathbf{Z}^{(k)} & \mathbf{Z}^{(k+1)} \\ \mathbf{Z}^{(3)} & \mathbf{Z}^{(4)} & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \mathbf{Z}^{(k-1)} & \mathbf{Z}^{(k)} & \dots & \dots & \mathbf{Z}^{(2k-3)} & \mathbf{Z}^{(2k-2)} \\ \mathbf{Z}^{(k)} & \mathbf{Z}^{(k+1)} & \dots & \dots & \mathbf{Z}^{(2k-2)} & \mathbf{Z}^{(2k-1)} \end{bmatrix}.$$

1.3.2. Circulant Matrices

In a symmetric circulant matrix, the number of distinct blocks is

$$m = \begin{cases} s + 1, & \text{if } k = 2s, \\ s, & \text{if } k = 2s - 1. \end{cases}$$

This is a special case of Toeplitz matrices with equal blocks on the diagonals, $\mathcal{A}_l = \{(i, j) : |i - j| = l - 1 \vee k - l + 1\}$, $l = 1, \dots, m$:

$$\mathbf{W}_X = \begin{bmatrix} \mathbf{Z}^{(1)} & \mathbf{Z}^{(2)} & \dots & \mathbf{Z}^{(l)} & \dots & \mathbf{Z}^{(3)} & \mathbf{Z}^{(2)} \\ \mathbf{Z}^{(2)} & \mathbf{Z}^{(1)} & \dots & \mathbf{Z}^{(l-1)} & \dots & \mathbf{Z}^{(4)} & \mathbf{Z}^{(3)} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \mathbf{Z}^{(3)} & \mathbf{Z}^{(4)} & \dots & \mathbf{Z}^{(m-1)} & \dots & \mathbf{Z}^{(1)} & \mathbf{Z}^{(2)} \\ \mathbf{Z}^{(2)} & \mathbf{Z}^{(3)} & \dots & \mathbf{Z}^{(m)} & \dots & \mathbf{Z}^{(2)} & \mathbf{Z}^{(1)} \end{bmatrix}.$$

The spectral analysis of random block matrices like Toeplitz or Hankel matrices helps to understand convergence for deep learning optimization; see, e.g., Xia et al. [8].

2. Main Result

Let $\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \dots, \mathbf{V}^{(m)}$ be Hermitian random matrices of size $n \times n$ whose elements are independent. Using the index sets introduced above (3), we define the following matrices:

$$\mathbf{Y}^{(ij)} = \mathbf{V}^{(l)}, \text{ for } (i, j) \in \mathcal{A}_l, \quad l = 0, \dots, m,$$

$$\mathbf{W}_Y = \sum_{i,j=1}^k \mathbf{E}^{(ij)} \otimes \mathbf{Y}^{(ij)} = \sum_{l=0}^m \sum_{(i,j) \in \mathcal{A}_l} \mathbf{E}^{(ij)} \otimes \mathbf{V}^{(l)}.$$

Assume that the random variables $Y_{pq}^{(ij)}$ with any fixed $i, j = 1 < \dots < k, i \leq j$, are independent for $p, q = 1, \dots, n, p \leq q$, and have the same first two moments as the random variables $X_{pq}^{(ij)}$; that is,

$$\mathbb{E}Y_{pq}^{(ij)} = 0, \text{ and } \mathbb{E}(Y_{pq}^{(ij)})^2 = (\sigma_{pq}^{(ij)})^2. \quad (5)$$

Denote the eigenvalues of matrix $\frac{1}{\sqrt{nk}} \mathbf{W}_X$ as $\lambda_1, \dots, \lambda_{nk}$ and the eigenvalues of $\frac{1}{\sqrt{nk}} \mathbf{W}_Y$ as $\mu_1, \dots, \mu_{nk}$. We define the empirical spectral distribution functions

$$F_{nX}(x) = \frac{1}{nk} \sum_{j=1}^{nk} \mathbb{I}\{\lambda_j < x\}$$

and

$$F_{n\mathbf{Y}}(x) = \frac{1}{nk} \sum_{j=1}^{nk} \mathbb{I}\{\mu_j < x\},$$

where $\mathbb{I}\{\cdot\}$ stands for the event indicator.

The empirical spectral distributions represent one of the main subjects of investigation in random matrix theory. We are interested in conditions regarding the distributions of $X_{rs}^{(ij)}$ ($X_{rs}^{(ij)}$) that would guarantee that the distributions $F_{n\mathbf{X}}(x)$ and $F_{n\mathbf{Y}}(x)$ become infinitely close in the limit of large n . We prove this closeness under Lindeberg's condition (see (7)). Lindeberg's condition shows that the contribution to the variance of the sum of n^2 random variables and the values of individual random variables exceeding the $\sqrt{n}$ level is negligibly small compared to n^2 .

This condition is the optimal moment condition in many limit theorems, both for sums of independent random variables (the central limit theorem for instance) and in random matrix theory (Wigner's semicircular law, as shown in [30], and the Marchenko–Pastur law for the empirical spectral distribution of sample covariance matrices; see [31]).

In what follows, for $(i, j) \in \mathcal{A}_l$, we shall write

$$\sigma_{pq}^{(ij)} = \sigma_{pq}^{(l)}.$$

Theorem 1. We assume that there exists a constant C_0 such that for all $n \geq 1$,

$$\frac{1}{(nk)^2} \sum_{l=1}^m \sum_{p,q=1}^n (\sigma_{pq}^{(l)})^2 \leq C_0. \quad (6)$$

Suppose that for all $i, j = 1, \dots, k$ for random matrices $\mathbf{X}^{(ij)}$ and $\mathbf{Y}^{(ij)}$, Lindeberg's condition is satisfied, i.e., for any $\tau > 0$,

$$L_n(\tau) := \max \left\{ \frac{1}{n^2} \sum_{r,s=1}^n \mathbb{E} (X_{rs}^{(ij)})^2 \mathbb{I}\{|X_{rs}^{(ij)}| > \tau\sqrt{n}\}, \right. \\ \left. \frac{1}{n^2} \sum_{r,s=1}^n \mathbb{E} (Y_{rs}^{(ij)})^2 \mathbb{I}\{|Y_{rs}^{(ij)}| > \tau\sqrt{n}\}, \right\} \xrightarrow{n \rightarrow \infty} 0. \quad (7)$$

Then

$$F_{n\mathbf{X}}(x) - F_{n\mathbf{Y}}(x) \xrightarrow{n \rightarrow \infty} 0 \text{ in probability.}$$

Remark 1. Note that if for some $\delta > 0$, there exists a constant $\mu_{2+\delta} > 0$ such that

$$\sup_{i,j,p,q \geq 1} \max \{ \mathbb{E} |Y_{pq}^{(i,j)}|^{2+\delta}, \mathbb{E} |X_{pq}^{(i,j)}|^{2+\delta} \} \leq \mu_{2+\delta}, \quad (8)$$

then Lindeberg's condition is satisfied. This follows from the Markov inequality. To find the limit distribution of $F_{n\mathbf{X}}(x)$, we can compute it for $F_{n\mathbf{Y}}(x)$, where $Y_{pq}^{(ij)}$ are Gaussian, for instance.

Remark 2. Figures 1–4 show histograms of the distribution of the eigenvalues of the Hankel (Figure 1) and Teuplitz (Figure 2) matrices with different distributions of block elements. We consider distributions with polynomially decreasing tails (stepped tails with decreasing order $|x|^{-k}$, where $k = 100b$ in Figures 1A and 2A and $k = 7/2$ in Figures 1B and 2B, respectively. Figures 3 and 4 show the histograms of the Teuplitz (Figure 3) and Hankel (Figure 4) matrices with block elements distributed according to Student's law with five degrees of freedom (Figures 3A and 4A) and with block elements distributed according to the normal law.

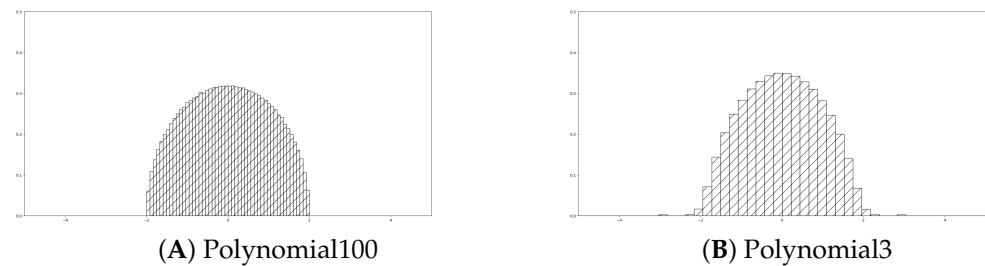


Figure 1. The histograms of 5×5 Hankel matrices with block entries distributed according to the polynomial densities $p(x) = \frac{C}{(1+|x|)^{100}}$ (A) and $p(x) = \frac{C}{(1+|x|)^2}$ (B).

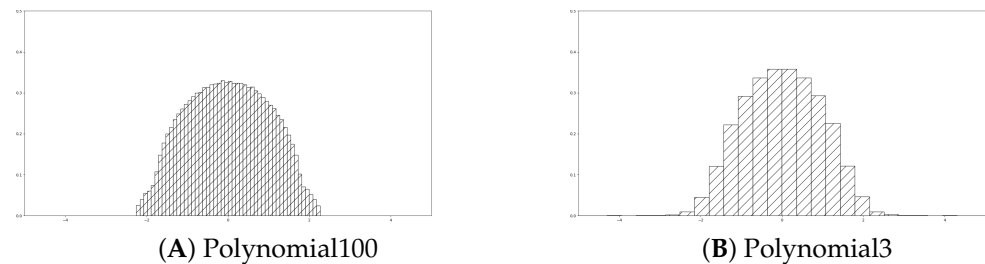


Figure 2. The histograms of 5×5 Toeplitz matrices with block entries distributed according to the polynomial densities $p(x) = \frac{C}{(1+|x|)^{100}}$ (A) and $p(x) = \frac{C}{(1+|x|)^2}$ (B).

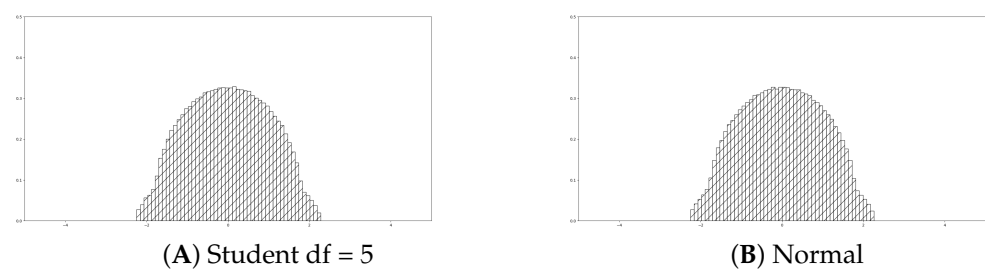


Figure 3. The histograms of 5×5 Toeplitz matrices with block entries distributed according to Student's distribution with $df = 5$ degrees of freedom (A) and a standard normal distribution (B).

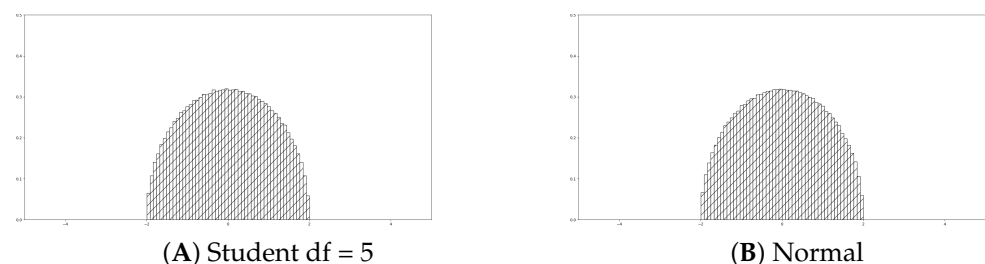


Figure 4. The histograms of 5×5 Hankel matrices with block entries distributed according to Student's distribution with $df = 5$ degrees of freedom (A) and a standard normal distribution (B).

It was proven in [5] that there exists a limiting distribution $G(x) = \lim_{n \rightarrow \infty} F_{nX}(x)$ for the i.i.d. matrix elements $Z^{(l)}$, for $l = 1, \dots, m$, and $\mathbb{E}|Z_{jk}^{(l)}|^4 < \infty$. Since we can choose matrices with identically distributed Gaussian elements as a sequence of matrices $Y^{(l)}$, we immediately obtain the following result as a consequence of the Theorem 1.

Theorem 2. Let $(\sigma_{pq}^{(i,j)})^2 = \sigma^2$. Suppose that for all $i, j = 1, \dots, k$, Lindeberg's condition for random variables $X_{pq}^{(ij)}$ and $Y_{pq}^{(ij)}$ is satisfied, i.e., for any $\tau > 0$,

$$L_n(\tau) := \frac{1}{n^2} \sum_{r,s=1}^n \mathbb{E}(X_{rs}^{ij})^2 \mathbb{I}\{|X_{rs}^{ij}|^2 > \tau\sqrt{n}\} \xrightarrow{n \rightarrow \infty} 0.$$

Then there exists a distribution function, $G(x)$, depending only on the structure of the block matrix, such that

$$\lim_{n \rightarrow \infty} F_n \mathbf{X}(x) = G(x). \quad (9)$$

The proof follows obviously from Theorem 1 above and Theorem 1 in [5].

Rectangular Blocks

Consider a block random matrix model with rectangular blocks. In this case, matrices $\mathbf{X}^{(ij)}$ satisfy the relation $\mathbf{X}^{(ij)} = (\mathbf{X}^{(ji)})^*$ (where, for any matrix $\mathbf{A}$, its Hermite conjugate $\overline{\mathbf{A}}^T$ is denoted by $\mathbf{A}^*$). The dimensions of the matrix $\mathbf{X}^{(ij)}$ are $n_i \times n_j$, respectively. Consider the class of generating matrices $\mathbf{H}_1, \dots, \mathbf{H}_k, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_m, \mathbf{Z}_1^*, \mathbf{Z}_2^*, \dots, \mathbf{Z}_m^*$, where the diagonal blocks $\mathbf{H}_1, \dots, \mathbf{H}_k$ are Hermitian matrices. The sets $\mathcal{A}_l$ and $\mathcal{A}_l^*$ are defined by the equations

$$\begin{aligned} \mathcal{A}_l &= \{(i, j) \mid i, j = 1, \dots, k : \mathbf{X}^{(ij)} = \mathbf{Z}_l\}, \\ \mathcal{A}_l^* &= \{(i, j) \mid i, j = 1, \dots, k : \mathbf{X}^{(ij)} = \mathbf{Z}_l^*\}. \end{aligned}$$

Clearly, $(i, j) \in \mathcal{A}_l$ if and only if $(j, i) \in \mathcal{A}_l^*$. We then have the following representation:

$$\mathbf{W}_\mathbf{X} = \sum_{i=1}^k \mathbf{e}_i \mathbf{e}_i^T \otimes \mathbf{H}_i + \sum_{l=1}^m \left(\sum_{(i,j) \in \mathcal{A}_l} \mathbf{e}_i \mathbf{e}_j^T \otimes \mathbf{Z}_l + \sum_{(i,j) \in \mathcal{A}_l^*} \mathbf{e}_i \mathbf{e}_j^T \otimes \mathbf{Z}_l^* \right). \quad (10)$$

Since we consider the case where the matrix $\mathbf{W}$ is Hermitian, i.e., $\mathbf{W}_\mathbf{X} = \mathbf{W}_\mathbf{X}^*$, we have a number of restrictions on the matrix block dimensions. For example, the matrix

$$\mathbf{W}_\mathbf{X} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{X}^{(12)} & \mathbf{X}^{(13)} \\ \mathbf{X}^{(12)*} & \mathbf{H}_2 & \mathbf{X}^{(23)} \\ \mathbf{X}^{(13)*} & \mathbf{X}^{(23)*} & \mathbf{H}_3 \end{bmatrix} \quad (11)$$

can be represented as

$$\begin{aligned} \mathbf{W}_\mathbf{X} &= (\mathbf{e}_1 \mathbf{e}_1^T \otimes \mathbf{H}_1 + \mathbf{e}_2 \mathbf{e}_2^T \otimes \mathbf{H}_2 + \mathbf{e}_3 \mathbf{e}_3^T \otimes \mathbf{H}_3) + (\mathbf{e}_1 \mathbf{e}_2^T \otimes \mathbf{X}^{(12)} \\ &\quad + (\mathbf{e}_2 \mathbf{e}_1^T \otimes \mathbf{X}^{(12)*} + (\mathbf{e}_1 \mathbf{e}_3^T \otimes \mathbf{X}^{(13)} + (\mathbf{e}_3 \mathbf{e}_1^T \otimes \mathbf{X}^{(13)*} + (\mathbf{e}_2 \mathbf{e}_3^T \otimes \mathbf{X}^{(23)} \\ &\quad + (\mathbf{e}_3 \mathbf{e}_2^T \otimes \mathbf{X}^{(23)*}). \end{aligned} \quad (12)$$

Matrices $\mathbf{H}^{(i)}$ are independent Hermitian matrices of order $N_i \times N_i$. Assume that the elements of $\mathbf{H}^{(i)} = (h_{pq}^{(i)})_{p,q=1}^{N_i}$ are independent (except for $\mathbf{H}^{(i)}$ being Hermitian) and

$$\mathbb{E} h_{pq}^{(i)} = 0, \mathbb{E} |h_{pq}^{(i)}|^2 = [\sigma_{pq}^{(i,0)}]^2 < \infty. \quad (13)$$

For matrices $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \dots, \mathbf{Z}^{(m)}$, we also assume that the elements are independent and

$$\mathbb{E} [\mathbf{Z}^{(l)}]_{pq} = 0, \mathbb{E} |[\mathbf{Z}^{(l)}]_{pq}|^2 = [\sigma_{pq}^{(l)}]^2 < \infty. \quad (14)$$

We denote

$$n = N_1 + N_2 + \dots + N_k. \quad (15)$$

Figures 5 and 6 show the histograms of random matrices in block form with rectangular blocks with different distributions of block elements. The figures illustrate the independence of the limiting spectral distribution of block random matrices with rectangular blocks from the block element distributions. In these notations, the matrix $\mathbf{W}_\mathbf{X}$ has dimension $n \times n$.

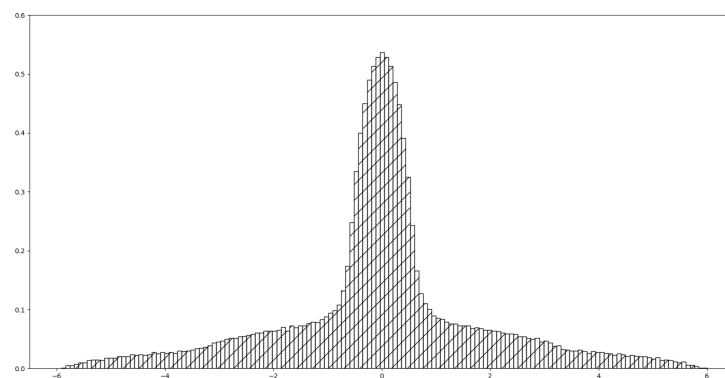


Figure 5. The histogram of 3×3 matrices with rectangular blocks $N_1 = 5000$, $N_2 = 3333$, $N_3 = 1666$. Student distribution with $df = 3$ degrees of freedom with different variances for different blocks.

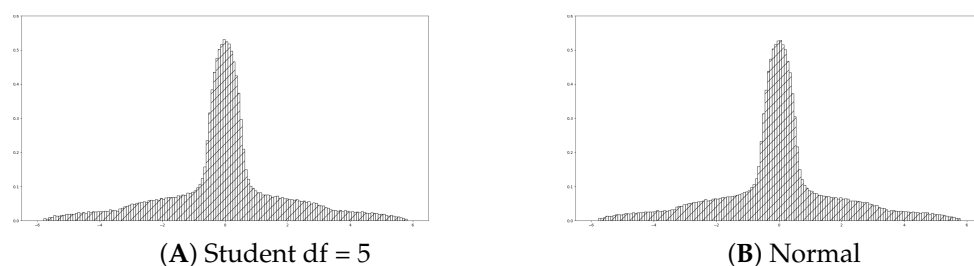


Figure 6. The histograms of 3×3 matrices with rectangular blocks: $N_1 = 5000$, $N_2 = 3333$, and $N_3 = 1666$. Student's distribution with $df = 5$ degrees of freedom (A) and normal distribution (B) with different variances for different blocks.

Now, let $\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(l)}$ and $\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)}$ be random matrices whose dimensions coincide with those of $\mathbf{H}^{(1)}, \dots, \mathbf{H}^{(k)}$ and $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \dots, \mathbf{Z}^{(m)}$ and whose elements are independent and have moments of the first two orders, coinciding with those of the elements of the above-mentioned matrices. Denote the matrix obtained from $\mathbf{W}_X$ by replacing blocks $\mathbf{H}_1, \dots, \mathbf{H}_k, \mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \dots, \mathbf{Z}^{(m)}$ with blocks $\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(l)}, \mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(m)}$, respectively, by $\mathbf{W}_Y$. We denote the eigenvalues of $\mathbf{W}_X$ by $\lambda_1, \lambda_2 \dots \lambda_n$ in decreasing order. We then have the empirical spectral distribution function

$$F_{nX}(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}\{\lambda_j \leq x\},$$

We denote the eigenvalues of matrix $\mathbf{W}_Y$ by $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, and again, we have the empirical spectral distribution function

$$F_{nY}(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}\{\mu_j \leq x\}.$$

Theorem 3. Assume that there exists a constant C_0 such that for all $n \geq 1$,

$$\frac{1}{(nk)^2} \sum_{l=1}^m \sum_{(i,j) \in \mathcal{A}_l} \sum_{p=1}^{N_i} \sum_{q=1}^{N_j} (\sigma_{pq}^{(i,j)})^2 + \frac{1}{n^2} \sum_{l=1}^k \sum_{p,q=1}^{N_l} \sigma_{pq}^{(l)} \leq C_0. \quad (16)$$

Suppose that for all $i, j = 1, \dots, k$ Lindeberg's condition is satisfied, i.e., for any $\tau > 0$,

$$L_n(\tau) := \frac{1}{n^2} \sum_{l=1}^m \sum_{(i,j) \in \mathcal{A}_l} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} \mathbb{E}(X_{rs}^{ij})^2 \mathbb{I}\{|X_{rs}^{ij}|^2 > \tau \sqrt{n}\} \\ + \frac{1}{n^2} \sum_{l=1}^k \sum_{p,q=1}^{n_l} \mathbb{E}[h_{pq}^{(l)}]^2 \mathbb{I}\{|h_{pq}^{(l)}| > \tau \sqrt{n}\} \xrightarrow{n \rightarrow \infty} 0.$$

Then

$$F_{n\mathbf{X}}(x) - F_{n\mathbf{Y}}(x) \xrightarrow{n \rightarrow \infty} 0 \text{ in probability.}$$

Remark 3. Figures 5 and 6 show the histograms of random matrices in block form with rectangular blocks with different distributions of block elements. The figures illustrate the independence of the limiting spectral distribution of block random matrices with rectangular blocks from the block element distributions.

For many block-type random matrices (both with square blocks and rectangular blocks), we know the limiting empirical spectral distributions in the case when the block elements are Gaussian values and the variances $[\sigma_{pq}^{(ij)}]^2$ satisfy certain conditions (e.g., they are all equal). Theorems 1 and 3 show that the same limit distributions will hold for blocks with arbitrary element distributions, as long as Lindeberg's conditions are satisfied. We can consider several examples.

We first consider two simple ones, the Hermitian matrix $\mathbf{X} = \frac{1}{\sqrt{n}}(X_{pq})_{p,q=1}^n$ and a rectangular matrix $\hat{\mathbf{X}}$ of dimensions $n \times m$, whose elements are independent. Consider the following block matrices:

$$\mathbf{W} = \begin{bmatrix} \mathbf{O} & \mathbf{X} & \mathbf{O} \\ \mathbf{X} & \mathbf{O} & \mathbf{X} \\ \mathbf{O} & \mathbf{X} & \mathbf{O} \end{bmatrix}, \quad \widehat{\mathbf{W}} = \begin{bmatrix} \mathbf{O} & \hat{\mathbf{X}} & \mathbf{O} \\ \hat{\mathbf{X}}^* & \mathbf{O} & \hat{\mathbf{X}} \\ \mathbf{O} & \hat{\mathbf{X}}^* & \mathbf{O} \end{bmatrix}, \quad (17)$$

It is well known that if the random variables X_{ij} are standard Gaussian and identically distributed, then the empirical spectral distribution function of matrix $\mathbf{X}$ is a standard semicircular law, i.e., the distribution density is

$$p(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{I}\{|x| \leq 2\}.$$

It is easy to see that

$$\det\{(\mathbf{W}_{\mathbf{X}} - z\mathbf{I})\} = z^n \det(2\mathbf{X}^2 - z^2\mathbf{I}). \quad (18)$$

It follows that the limiting empirical spectral distribution is a mixture of a $\frac{1}{3}$ atom at zero and a distribution with density

$$p_1(x) = \frac{1}{4\pi} \sqrt{8 - x^2} \mathbb{I}\{|x| \leq 2\sqrt{2}\}$$

and mass $\frac{2}{3}$, so that the density of the distribution can be written as

$$\tilde{p}(x) = \frac{1}{3} \delta_0(x) + \frac{2}{3} p_1(x),$$

where $\delta_0(x)$ is the Dirac delta function. From Theorem 1, if for random variables X_{pq} , $p, q = 1, \dots, n$, we have $\mathbb{E}[X_{pq}]^2 = 1$ and Lindeberg's condition is satisfied, then the empirical spectral distribution function of $\mathbf{W}_{\mathbf{X}}$ has a density $\tilde{p}(x)$. (Matrices of this type have also been considered in [12].) In a similar way, we can consider the case of a rectangular

block. Assume that $\lim_{n \rightarrow \infty} \frac{n}{m} = y \in (0, 1)$. Applying the formula for the determinant of a block matrix, we obtain

$$\det\{\widehat{\mathbf{W}} - z\mathbf{I}\} = (-z)^n \det\{2\widehat{\mathbf{X}}^* \widehat{\mathbf{X}} - z^2 \mathbf{I}\}^{-1} \quad (19)$$

From here, one easily sees that the eigenvalues of $\mathbf{W}_{\widehat{\mathbf{X}}}$ are zero and of multiplicity $2m - n$ and $\pm s_j$, $j = 1, \dots, n$, where s_j are singular numbers of $\widehat{\mathbf{X}}$. For example, in the case where X_{ij} is Gaussian with variance, the relation

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{j=1}^n (\mathbb{I}\{s_j \leq x\} + \mathbb{I}\{-s_j \leq x\}) \rightarrow G_y(x), \text{ at } n \rightarrow \infty, \quad (20)$$

where $G_y(x)$ is the function of the symmetrized Marchenko–Pastur distribution with the parameter y , i.e., a distribution with density

$$g_y(x) = \frac{1}{2\pi y x} \sqrt{(x^2 - a^2)(b^2 - a^2) \mathbb{I}\{a \leq |x| \leq b\}},$$

where $a = (1 - \sqrt{y})^2$, $b = (1 + \sqrt{y})^2$. From Theorem 3, we obtain the following: if $\mathbb{E}[\widehat{X}_{ij}]^2 = 1$ and Lindeberg's condition is satisfied, then the limiting distribution of the spectral distribution function of the matrix $\widehat{\mathbf{W}}$ will have the form

$$\widehat{p}(x) = \frac{2-y}{2+y} \delta_0(x) + \frac{2y}{2+y} \frac{1}{\sqrt{2}} g_y(x/\sqrt{2}).$$

Results similar to Theorem 3 were obtained in [32]. In contrast to our paper, in [32], the proof was carried out using the method of moments, and instead of condition (16), the condition

$$\sup_{j,l,p,q} (\sigma_{pq}^{(jl)})^2 \leq C_0 \quad (21)$$

was considered.

Furthermore, in [32], it was assumed that all blocks $\mathbf{X}^{(ij)}$, $i, j = 1, \dots, k$, were independent, which rules out the case of coincident blocks, such as in the case of Hankel or Töplitz matrices. Finally, in [32], it was assumed that $\sigma_{pq}^{(ij)} = \sigma_{pq}$, i.e., the variance within blocks is the same. One cannot, for example, apply the results of [32] to the matrix $\mathbf{W} = \begin{bmatrix} & \mathbf{X} \\ \mathbf{X} & \end{bmatrix}$, where $\mathbf{X}$ is a Wigner matrix (Hermitian matrices with independent elements (to Hermitian accuracy) with equal variance) whose limiting spectral distribution is a semicircular law (cf. Corollary 2 in [32]).

Based on Theorems 1 and 3 and the results of Theorems 2 and 4 in [12], we can formulate some theorems on the convergence of the empirical spectral distribution functions of block-type random matrices. For this purpose, we need one more notation. We denote the covariance of the random elements of the matrices $\mathbf{X}^{(ij)}$ and $\mathbf{X}^{(pq)}$ by $\sigma_{i,j;p,q}$. By virtue of our agreement that the matrices $\mathbf{X}^{(ij)}$ and $\mathbf{X}^{(pq)}$ are either independent or coincident, the values of the function $\sigma_{i,j;p,q}$ are either 0 or 1. It is obvious that

$$\sigma_{i,j;p,q} = \sigma_{p,q;i,j}.$$

Let $\mathbb{M}_k(\mathbb{C})$ be a set of $k \times k$ matrices with complex elements. We define the mapping $\eta : \mathbb{M}_k(\mathbb{C}) \rightarrow \mathbb{M}_k(\mathbb{C})$ as follows. For a matrix $\mathbf{D} = (d_{ij})_{i,j=1}^k \in \mathbb{M}_k(\mathbb{C})$, we have $\eta(\mathbf{D}) = (\eta(\mathbf{D})_{ij})_{i,j=1}^k$:

$$[\eta(\mathbf{D})]_{ij} = \frac{1}{k} \sum_{p,q=1}^k \sigma(i, j; p, q) d_{pq}. \quad (22)$$

Theorem 4. Suppose that the random variables $X_{pq}^{(ij)}$, where $i, j = 1, \dots, k$ and $p, q = 1, \dots, n$, satisfy the following conditions:

$$(1) \mathbb{E}X_{pq}^{(ij)} = 0 \text{ and } \mathbb{E}|X_{pq}^{(ij)}|^2 = \sigma^2;$$

(2) Lindeberg's condition: for any $\tau > 0$, we have

$$L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^k \sum_{p,q=1}^n \mathbb{E}|X_{pq}^{(ij)}|^2 \mathbb{I}\{|X_{pq}^{(ij)}| > \tau\sqrt{n}\} \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Then there exists a distribution function $F(x)$ with a Stieltjes transform $S(z)$ that satisfies the equality $S(z) = \mathfrak{G}(z)/k$, where the function $\mathfrak{G}(z)$ is an analytic function taking values in the space of square matrices of dimensions $k \times k$, defined in the upper complex half-plane and uniquely defined by the properties

$$\lim_{|z| \rightarrow \infty, \operatorname{Im} z > 0} z\mathfrak{G}(z) = \mathbf{I}_k$$

and

$$z\mathfrak{G}(z) = \mathbf{I}_k + \eta(\mathfrak{G}(z))\mathfrak{G}(z).$$

3. Proof of the Main Result

We only need to show the convergence of the corresponding Stieltjes transforms in any subset of the upper complex half-plane with a non-empty interior. Consider the resolvent $\mathbf{R}_{\mathbf{A}}(z)$ of a symmetric matrix $\mathbf{A}$. This is defined for all $z = u + iv$ with $v > 0$. We only need to prove that in some region $\mathcal{G} \subset \mathbb{C}_+$ with a non-empty interior, there is the convergence

$$\frac{1}{nk} \operatorname{Tr} \mathbf{R}_{\mathbf{W}_X}(z) - \frac{1}{nk} \operatorname{Tr} \mathbf{R}_{\mathbf{W}_Y} \rightarrow 0 \text{ for } n \rightarrow \infty \quad (23)$$

in probability.

We shall divide the proof into three parts. First, we show that we can replace random variables with so-called truncated random variables. Then, we prove that for truncated quantities, the difference between the expectations of the Stieltjes transformations of the matrices $\mathbf{W}_X$ and $\mathbf{W}_Y$ tends to 0. Finally, we show that the variance of the Stieltjes transformations tends to 0 (Girko's lemma). The convergence of expectations and the convergence of variances to zero entail the convergence of probabilities.

3.1. Truncation

As a first step in the proof, we reduce the problem to truncated random variables. For some $\tau > 0$, we introduce the quantities

$$\tilde{X}_{pq}^{(l)} = X_{pq}^{(l)} \mathbb{I}\{|X_{pq}^{(l)}| \leq \tau\sqrt{n}\}.$$

Next, consider their centered version

$$\hat{X}_{pq}^{(l)} = \tilde{X}_{pq}^{(l)} - \mathbb{E}\tilde{X}_{pq}^{(l)}$$

and the normalized version

$$\check{X}_{pq}^{(l)} = \frac{\sigma_{pq}^{(l)}}{\widehat{\sigma}_{pq}^{(l)}} \widehat{X}_{pq}^{(l)},$$

where $[\widehat{\sigma}_{pq}^{(l)}]^2 = \mathbb{E}[\widehat{X}_{pq}^{(l)}]^2$. Note that

$$\mathbb{E}\check{X}_{pq}^{(l)} = 0, \quad \mathbb{E}[\check{X}_{pq}^{(l)}]^2 = [\sigma_{pq}^{(l)}]^2.$$

Consider the matrices $\widetilde{\mathbf{W}}_{\mathbf{X}}$, $\widehat{\mathbf{W}}_{\mathbf{X}}$, and $\check{\mathbf{W}}_{\mathbf{X}}$ obtained from $\mathbf{W}_{\mathbf{X}}$ by replacing $X_{pq}^{(l)}$ with $\widetilde{X}_{pq}^{(l)}$, $\widehat{X}_{pq}^{(l)}$, and $\check{X}_{pq}^{(l)}$, respectively. Also, consider the resolvent matrices $\widetilde{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}}(z)$, $\widehat{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}}(z)$, and $\check{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}}(z)$.

Lemma 1. *Under the conditions of Theorem 1, we have*

$$\lim_{n \rightarrow \infty} \left(\frac{1}{nk} \text{Tr} \check{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}} - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}} \right) = 0 \text{ in probability.}$$

Proof. We start by estimating the quantity

$$B_1 := \frac{1}{nk} \text{Tr} \widetilde{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}}(z) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z).$$

The following inequality holds:

$$\mathbb{E} \|\mathbf{W}_{\mathbf{X}} - \widetilde{\mathbf{W}}_{\mathbf{X}}\|_2^2 \leq \frac{1}{nk} \sum_{l=1}^m \sum_{(i,j) \in \mathcal{A}_l} \sum_{p,q=1}^n \mathbb{E} |X_{pq}^{(ij)}|^2 \mathbb{I}\{|X_{pq}^{(ij)}| > \tau\sqrt{n}\}. \quad (24)$$

Then, applying Cauchy's inequality, it is easy to obtain that

$$\mathbb{E}|B_1| \leq \frac{1}{nk} \mathbb{E} \left(\sum_{l,r=1}^{nk} \left| \sum_{j=1}^{nk} [\mathbf{R}(z)]_{jl} [\widetilde{\mathbf{R}}(z)]_{jr} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{l,r=1}^{kn} \mathbb{E} |W_{lr} - \widetilde{W}_{lr}|^2 \right)^{\frac{1}{2}}.$$

It is then not hard to see that

$$\sum_{l,r=1}^{nk} \left| \sum_{j=1}^{nk} [\mathbf{R}(z)]_{jl} [\widetilde{\mathbf{R}}(z)]_{jr} \right|^2 = \sum_{l,r=1}^{nk} |[\mathbf{R}(z) \widetilde{\mathbf{R}}(z)]_{lr}|^2 = \|\mathbf{R}(z) \widetilde{\mathbf{R}}(z)\|_2^2 \leq \frac{nk}{v^4}. \quad (25)$$

Combining inequalities (24) and (25), we obtain the following:

$$\mathbb{E}|B_1| \leq \left(\frac{C}{n^2} \sum_{l=1}^m \sum_{p,q=1}^n \mathbb{E} (X_{pq}^{(l)})^2 \mathbb{I}\{|X_{pq}^{(l)}| > \tau\sqrt{n}\} \right)^{\frac{1}{2}} = \frac{1}{v^2} L_n^{\frac{1}{2}}(\tau). \quad (26)$$

We obtain a similar estimate for the quantity

$$\mathbb{E}|B_2| := \mathbb{E} \left| \frac{1}{nk} \text{Tr} \widetilde{\mathbf{R}}(z)_{\mathbf{X}} - \frac{1}{nk} \text{Tr} \widehat{\mathbf{R}}_{\mathbf{X}}(z) \right| \leq \frac{1}{v^2} L_n^{\frac{1}{2}}(\tau). \quad (27)$$

Let us now estimate the value

$$B_3 := \frac{1}{nk} \text{Tr} \check{\mathbf{R}}_{\mathbf{X}}(z) - \frac{1}{nk} \text{Tr} \widehat{\mathbf{R}}_{\mathbf{X}}(z). \quad (28)$$

We have

$$\begin{aligned} \mathbb{E}|B_3| &\leq \frac{1}{nk} \mathbb{E} \|\widehat{\mathbf{R}}_{\mathbf{X}}(z) \check{\mathbf{R}}_{\mathbf{X}}(z) \check{\mathbf{R}}_{\mathbf{X}}(z)\|_2, \\ \mathbb{E} \|\check{\mathbf{W}} - \widehat{\mathbf{W}}\|_2 &\leq \frac{1}{\sqrt{nk}v^2} \mathbb{E} \|\check{\mathbf{W}} - \widehat{\mathbf{W}}\|_2 \\ &= \frac{1}{nk v^2} \mathbb{E}^{\frac{1}{2}} \left(\sum_{l=1}^m \sum_{(ij) \in \mathcal{A}_l} \sum_{p,q=1}^n |\widehat{X}_{pq}^{(ij)} - \check{X}_{pq}^{(ij)}|^2 \right). \end{aligned} \quad (29)$$

Obviously,

$$\begin{aligned} \sum_{l=1}^m \sum_{(ij) \in \mathcal{A}_l} \sum_{p,q=1}^m \mathbb{E} |\widehat{X}_{pq}^{(ij)} - \check{X}_{pq}^{(ij)}|^2 &\leq \sum_{l=1}^m \sum_{(ij) \in \mathcal{A}_l} \sum_{p,q=1}^m |\widehat{\sigma}_{pq}^{(ij)}|^2 - (\sigma_{pq}^{(ij)})^2 \\ &\leq C(nk)^2 L_n(\tau). \end{aligned}$$

Substituting the last estimate into (29), we obtain

$$\mathbb{E}|B_3| \leq \frac{C}{v^2} L_n^{\frac{1}{2}}(\tau),$$

and therefore,

$$\mathbb{E} \left| \frac{1}{nk} \text{Tr} \check{\mathbf{R}}_{\mathbf{W}_{\mathbf{X}}}(z) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z) \right| \leq \frac{C}{v^2} L_n^{\frac{1}{2}}(\tau).$$

Lemma 1 has been proven. $\square$

Remark 4. Since the function $L_n(\tau)$ at fixed τ is monotone on τ and $L_n(\tau) \rightarrow 0$ at $n \rightarrow \infty$ for any fixed τ , there exists a sequence τ_n such that $\tau_n \rightarrow 0$ and $L_n(\tau_n) \rightarrow 0$ at $n \rightarrow \infty$. In what follows, we write $\tau = \tau_n \rightarrow 0$ such that $L_n(\tau_n) \rightarrow 0$ as $n \rightarrow \infty$. Without loss of generality, we can assume that the random variables $X_{pq}^{(ij)}$, $i, j = 1, \dots, k$; $p, q = 1, \dots, n$, and $Y_{pq}^{(ij)}$, $i, j = 1, \dots, k$; $p, q = 1, \dots, n$, satisfy the condition

$$\max\{|Y_{pq}^{(ij)}|, |X_{pq}^{(ij)}|\} \leq \tau \sqrt{n}. \quad (30)$$

3.2. Special Representation for the Difference Between Two Resolvents

For any $\alpha \in [0, \pi/2]$, consider the random matrix

$$\mathbf{Z}^{(l)}(\alpha) = \mathbf{X}_l \cos \alpha + \mathbf{Y}_l \sin \alpha.$$

We denote

$$\mathbf{W}_{\mathbf{X}}(\alpha) := \frac{1}{\sqrt{nk}} \sum_{l=1}^m \left(\sum_{(i,j) \in \mathcal{A}_l} \mathbf{E}^{(i,j)} \right) \otimes \mathbf{Z}^{(l)}(\alpha), \quad \mathbf{R}_{\mathbf{X}}(z, \alpha) = (\mathbf{W}(\alpha) - z\mathbf{I})^{-1}.$$

In what follows, we denote the elements of $\mathbf{W}_{\mathbf{X}}(\alpha) = (W_{jk})$ and $\mathbf{Z}^{(l)}(\alpha) = (Z_{jk}^{(l)})$ simply by $\mathbf{W}$ and $\mathbf{Z}^{(l)}$, omitting α and $\mathbf{X}$ from notations (unless this is ambiguous). Obviously, we have

$$\begin{aligned} \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{Y}}}(z) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z) &= \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z, \frac{\pi}{2}) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z, 0) \\ &= \frac{1}{nk} \int_0^{\frac{\pi}{2}} \frac{\partial \text{Tr} \mathbf{R}_{\mathbf{W}_{\mathbf{X}}}(z, \alpha)}{\partial \alpha} d\alpha = \frac{1}{nk} \sum_{j=1}^{nk} \int_0^{\frac{\pi}{2}} \frac{\partial [\mathbf{R}_{\mathbf{W}_{\mathbf{X}}}]_{jj}(z, \alpha)}{\partial \alpha} d\alpha. \end{aligned} \quad (31)$$

We can now write

$$\frac{\partial[\mathbf{R}_{\mathbf{W}_X}]_{jj}(z, \alpha)}{\partial \alpha} = \sum_{l=1}^m \sum_{p,q=1}^n \frac{\partial[\mathbf{R}_{\mathbf{W}_X}]_{jj}(z, \alpha)}{\partial Z_{pq}^{(l)}} \frac{\partial Z_{pq}^{(l)}}{\partial \alpha}.$$

Note that for any invertible matrix $\mathbf{A}$, the following differentiation formula is valid:

$$\frac{\partial \mathbf{A}_{jk}^{-1}}{\partial A_{pq}} = -[\mathbf{A}^{-1}]_{jp} [\mathbf{A}^{-1}]_{qk} \quad (32)$$

(see, for example, [33]). Applying (32) to the resolvent $\mathbf{R}_{\mathbf{W}_X}(z, \alpha)$, we obtain

$$\frac{\partial[\mathbf{R}_{\mathbf{W}_X}]_{jj}(z, \alpha)}{\partial Z_{pq}^{(l)}} = - \sum_{(r,s) \in \mathcal{A}_l} (2 - \delta_{pq}) [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(r-1)n+p} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(s-1)n+q},$$

where δ_{pq} stands for the Kronecker symbol. Also, note that

$$\frac{\partial Z_{pq}^{(l)}}{\partial \alpha} = -X_{pq}^{(l)} \sin \alpha + Y_{pq}^{(l)} \cos \alpha =: \tilde{Z}_{pq}^{(l)}.$$

Summing up the above formulas, we can write

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_Y}(z) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_X}(z) \right) \\ &= - \frac{1}{nk\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n (2 - \delta_{pq}) \int_0^{\frac{\pi}{2}} \mathbb{E} \tilde{Z}_{pq}^{(l)}(\alpha) \\ & \times \sum_{(r,s) \in \mathcal{A}_l} (2 - \delta_{pq}) [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(r-1)n+p} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(s-1)n+q} d\alpha. \end{aligned} \quad (33)$$

In what follows, we are going to estimate the values

$$D_{pq}^{(l,r,s)} = \mathbb{E} \tilde{Z}_{pq}^{(l)} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(r-1)n+p} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(s-1)n+q}$$

for $l = 1, \dots, m; r, s \in \mathcal{A}_l$ and $p, q = 1, \dots, n$. We introduce the matrices $\mathbf{Z}^{(l,p,q)} = \mathbf{Z}^{(l)} - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} (\mathbf{E}^{(p,q)} + \mathbf{E}^{(q,p)})$ (i.e., the matrix $\mathbf{Z}^{(l,p,q)}$ has zeros instead of each $Z_{pq}^{(l)}$). In the same way, we define the matrices $\mathbf{W}^{(l,p,q)}(z, \alpha)$ and $\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)$. The simple resolvent equality $\mathbf{R}_{\mathbf{A}_1} - \mathbf{R}_{\mathbf{A}_2} = \mathbf{R}_{\mathbf{A}_1} (\mathbf{A}_2 - \mathbf{A}_1) \mathbf{R}_{\mathbf{A}_2}$ shows that

$$\mathbf{R}_{\mathbf{W}_X}(z, \alpha) = \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) - \mathbf{R}_{\mathbf{W}_X}(z, \alpha) (\mathbf{W} - \mathbf{W}^{(l,p,q)}(z, \alpha)) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha). \quad (34)$$

Let $\mathbf{u}_j, j = 1, \dots, n$, denote the standard base vectors (column vectors) of the space $\mathbb{R}^n$. It is easy to see that we can then write

$$\mathbf{W}(\alpha) - \mathbf{W}^{(l,p,q)}(\alpha) = \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} \sum_{r,s \in \mathcal{A}_l} \mathbf{E}^{(r,s)} \otimes (\mathbf{u}_p \mathbf{u}_q^T + \mathbf{u}_q \mathbf{u}_p^T)$$

Now, let $\mathbf{v}_j, j = 1, \dots, kn$, be the corresponding basis vectors for $\mathbb{R}^{kn}$. Then we can write

$$\mathbf{E}^{(r,s)} \otimes (\mathbf{u}_p \mathbf{u}_q^T + \mathbf{u}_q \mathbf{u}_p^T) = \mathbf{v}_{(r-1)n+p} \mathbf{v}_{(s-1)n+q}^T + \mathbf{v}_{(r-1)n+q} \mathbf{v}_{(s-1)n+p}^T. \quad (35)$$

And, further,

$$\begin{aligned} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r-1)n+p} &= [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r-1)n+p} - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} \sum_{r_1, s_1 \in \mathcal{A}_l} \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}(z, \alpha) \\ &\quad \times (\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p}, \\ [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (s-1)n+q} &= \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)_{j, (s-1)n+q} - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} \sum_{r, s \in \mathcal{A}_l} \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}(z, \alpha) \\ &\quad \times (\mathbf{v}_{(r-1)n+p} \mathbf{v}_{(s-1)n+q}^T + \mathbf{v}_{(r-1)n+q} \mathbf{v}_{(s-1)n+p}^T) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(s-1)n+q}. \end{aligned} \quad (36)$$

Now applying (34) to the matrix $\mathbf{R}_{\mathbf{W}_X}$ on the right-hand side of (36), we arrive at the following equality:

$$\begin{aligned} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r-1)n+p} &= [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r-1)n+p} \\ &\quad - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} \sum_{r_1, s_1 \in \mathcal{A}_l} \left(\mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) (\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T) \right. \\ &\quad \times \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p} \Big) + \frac{1}{nk} (Z_{pq}^{(l)})^2 [T_1^{(l,p,q)}]_{j,r,s}, \\ [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (s-1)n+q} &= [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (s-1)n+q} \\ &\quad - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} \sum_{r_1, s_1 \in \mathcal{A}_l} \left(\mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}(z, \alpha) (\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T) \right. \\ &\quad \times \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(s-1)n+q} \Big) + \frac{1}{nk} (Z_{pq}^{(l)})^2 [T_2^{(l,p,q)}]_{j,r,s}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} [T_1^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}(z, \alpha) \left(\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T \right) \\ &\quad \times \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_2-1)n+p} \mathbf{v}_{(s_2-1)n+q}^T + \mathbf{v}_{(r_2-1)n+q} \mathbf{v}_{(s_2-1)n+p}^T \right) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(s-1)n+q}, \\ [T_2^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}(z, \alpha) \left(\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T \right) \\ &\quad \times \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_2-1)n+p} \mathbf{v}_{(s_2-1)n+q}^T + \mathbf{v}_{(r_2-1)n+q} \mathbf{v}_{(s_2-1)n+p}^T \right) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p}. \end{aligned}$$

Note that

$$\mathbb{E} \tilde{Z}^{(l)}(\alpha) Z_{pq}^{(l)}(\alpha) = \mathbb{E}((X_{pq}^{(l)})^2 - (Y_{pq}^{(l)})^2) \sin \alpha \cos \alpha = 0.$$

Through the independence of the quantities $X_{pq}^{(l)}$ and $Y_{pq}^{(l)}$ and the matrices $\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)$, we obtain

$$\begin{aligned} \mathbb{E} \tilde{Z}^{(l)}(\alpha) [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (s-1)n+q} &= 0, \\ \mathbb{E} \tilde{Z}^{(l)}(\alpha) Z_{pq}^{(l)}(\alpha) \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \\ &\quad \times (\mathbf{v}_{(r-1)n+p} \mathbf{v}_{(s-1)n+q}^T + \mathbf{v}_{(r-1)n+q} \mathbf{v}_{(s-1)n+p}^T) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p} = 0. \end{aligned} \quad (38)$$

Equalities (33), (36), and (38) imply that

$$\mathbb{E} \left(\frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_Y}(z) - \frac{1}{nk} \text{Tr} \mathbf{R}_{\mathbf{W}_X}(z) \right) = \int_0^{\frac{\pi}{2}} (Q_1 + \dots + Q_6) d\alpha, \quad (39)$$

where

$$\begin{aligned}
Q_1 &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^2 \\
&\quad \times \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T \right) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p} \\
&\quad \times \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_2-1)n+p} \mathbf{v}_{(s_2-1)n+q}^T + \mathbf{v}_{(r_2-1)n+q} \mathbf{v}_{(s_2-1)n+p}^T \right) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(s-1)n+q}, \\
Q_2 &= -\frac{1}{(nk)^3} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^3 \\
&\quad \times \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T \right) \\
&\quad \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(r-1)n+p} [T_2^{(l,p,q)}]_{j,r,s}, \\
Q_3 &= -\frac{1}{(nk)^3} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^3 \\
&\quad \times \sum_{r,s \in \mathcal{A}_l} (2 - \delta_{pq}) [T_1^{(l,p,q)}]_{j,r,s} \\
&\quad \times \mathbf{v}_j^T \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \left(\mathbf{v}_{(r_1-1)n+p} \mathbf{v}_{(s_1-1)n+q}^T + \mathbf{v}_{(r_1-1)n+q} \mathbf{v}_{(s_1-1)n+p}^T \right) \mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha) \mathbf{v}_{(s-1)n+q}, \\
Q_4 &= -\frac{1}{(nk)^3\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^4 \\
&\quad \times \sum_{(r,s) \in \mathcal{A}_l} (2 - \delta_{pq}) [T_1^{(l,p,q)}]_{j,r,s} [T_2^{(l,p,q)}]_{j,r,s}, \\
Q_5 &= \frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^2 \\
&\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r-1)n+p} [T_2^{(l,p,q)}]_{j,r,s}, \\
Q_6 &= \frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n (2 - \delta_{pq}) \mathbb{E} \tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^2 \\
&\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(s-1)n+q} [T_1^{(l,p,q)}]_{j,r,s}.
\end{aligned}$$

3.3. Estimations of Quantities Q_1 – Q_6

3.3.1. Estimation of Q_1

We represent Q_1 in the form

$$Q_1 = Q_{11} + \cdots + Q_{14},$$

where

$$\begin{aligned}
 Q_{11} &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^2 \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+p} \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+q, (s-1)n+q}, \\
 Q_{12} &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^2 \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+p} \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+p, (s-1)n+q}, \\
 Q_{13} &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^2 \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+p, (r-1)n+p} \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+q, (s-1)n+q}, \\
 Q_{14} &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^2 \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+p, (r-1)n+p} \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+p, (s-1)n+q}
 \end{aligned}$$

There is an obvious inequality for $v = 2, 3, 4$:

$$\frac{1}{(\sqrt{n})^{v-1}} \mathbb{E} |\tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^v| \leq C \tau^{v-1} (\sigma_{pq}^{(l)})^2. \quad (40)$$

This follows from condition (30).

All values of Q_{1v} , $v = 1, \dots, 4$, are estimated similarly. Let us first consider the estimation of Q_{11} . Note that the number of elements in sets $\mathcal{A}_l$ for $l = 1, \dots, m$ is independent of n and does not exceed k^2 . Given the independence of $X_{pq}^{(l)}$, $Y_{pq}^{(l)}$, and the matrices $\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}$, we can write

$$\begin{aligned}
 Q_{11} &= -\frac{1}{(nk)^2\sqrt{nk}} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (2 - \delta_{pq}) \mathbb{E} \tilde{Z}_{pq}^{(l)} (Z_{pq}^{(l)})^2 \\
 &\quad \times \mathbb{E} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+p} \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+q, (s-1)n+q}.
 \end{aligned}$$

Applying inequality (40), we get

$$\begin{aligned}
 |Q_{11}| &\leq C \tau \frac{1}{(nk)^2} \sum_{j=1}^{nk} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \sum_{r_2, s_2 \in \mathcal{A}_l} (\sigma_{pq}^{(l)})^2 \\
 &\quad \times \mathbb{E} \left| [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+p} \right. \\
 &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j, (r_2-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+q, (s-1)n+q} \Big|
 \end{aligned}$$

The following inequality is valid

$$\begin{aligned} & \sum_{j=1}^{nk} \left| [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r_2-1)n+p} \right| \\ & \leq \left(\sum_{j=1}^{nk} |[\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r_1-1)n+p}|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{nk} |[\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r_2-1)n+p}|^2 \right)^{\frac{1}{2}} \leq \frac{1}{v^2} \end{aligned} \quad (41)$$

Here, we apply Cauchy's inequality and use the fact that for any resolvent matrix $\mathbf{R}_A = (\mathbf{A} - z\mathbf{I})^{-1}$ constructed by a symmetric real or Hermite matrix $\mathbf{A} = (A_{ij})_{i,j=1}^n$, the operator norm at the point $z = u + iv$ does not exceed $\|\mathbf{R}_A\| \leq \frac{1}{v}$, and hence the sum of squares of the elements of any row does not exceed $\frac{1}{v^2}$, i.e., for any $i = 1, \dots, n$,

$$\sum_{j=1}^n |[\mathbf{R}_A]_{ij}|^2 \leq \frac{1}{v^2}. \quad (42)$$

In addition,

$$\max\{ |[\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q,(r-1)n+p}|, |[\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_2-1)n+q,(s-1)n+q}| \} \leq \frac{1}{v}. \quad (43)$$

Given these inequalities, we arrive at the following estimation:

$$|Q_{11}| \leq \frac{C\tau}{v^4} \frac{1}{(nk)^2} \sum_{l=1}^m \sum_{p,q=1}^n (\sigma_{pq}^{(l)})^2,$$

Similarly, we obtain estimates for $\nu = 2, 3, 4$:

$$|Q_{1\nu}| \leq C\tau v^{-4} \frac{1}{(nk)^2} \sum_{l=1}^m \sum_{p,q=1}^n (\sigma_{pq}^{(l)})^2. \quad (44)$$

Hence, from condition (16),

$$|Q_1| \leq \frac{C\tau}{v^4}. \quad (45)$$

The estimates of Q_2, Q_3 , and Q_4 are slightly different due to the dependence of the values $X_{pq}^{(l)}$ and $Y_{pq}^{(l)}$ and the matrices $\mathbf{R}_{\mathbf{W}_X}$. But this problem is easily circumvented. All estimates associated with the elements of the resolvent are fulfilled uniformly over the entire space where the random variables $X_{pq}^{(l)}$ and $Y_{pq}^{(l)}$ are defined.

3.3.2. Estimation of Q_2

To estimate Q_2 , we note that there exists a constant C such that

$$|[T_2^{(l,p,q)}]_{j,r,s}| \leq Cv^{-3} \sum_{r_1 \in \mathcal{A}_l} |[\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(r_1-1)n+p}|. \quad (46)$$

Applying this inequality, we obtain

$$\begin{aligned} |Q_2| & \leq C \frac{1}{v^4 (nk)^3} \sum_{l=1}^m \sum_{p,q=1}^n \sum_{(r,s) \in \mathcal{A}_l} \sum_{r_1, s_1 \in \mathcal{A}_l} \mathbb{E} |\tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^3| \\ & \quad \times \sum_{j=1}^{nk} |[\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{j,(r_1-1)n+p}| |[\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j,(r-1)n+p}|, \end{aligned} \quad (47)$$

Using inequalities (42) and (43), we obtain

$$|Q_2| \leq C \frac{1}{v^6 (nk)^3} \sum_{l=1}^m \sum_{p,q=1}^n \mathbb{E} |\tilde{Z}^{(l)}(\alpha) (Z_{pq}^{(l)})^3|. \quad (48)$$

Inequalities (40) and (16) and the last inequality together imply that

$$|Q_2| \leq C \frac{\tau^2}{v^5}. \quad (49)$$

3.3.3. Estimation of Q_3

Let us rewrite the definition of the quantity $[T_1^{l,p,q}]_{j,r,s}$ taking into account equality (35):

$$[T_1^{(l,p,q)}]_{jrs} = [T_{11}^{(l,p,q)}]_{jrs} + \dots + [T_{14}^{(l,p,q)}]_{jrs},$$

where

$$\begin{aligned} [T_{11}^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+p} \\ &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s-1)n+q, (s-1)n+q}, \\ [T_{12}^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+p} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+q, (r-1)n+q} \\ &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s-1)n+p, (s-1)n+q}, \\ [T_{13}^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+p, (r-1)n+p} \\ &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s-1)n+q, (s-1)n+q}, \\ [T_{14}^{(l,p,q)}]_{j,r,s} &= \sum_{r_1, s_1 \in \mathcal{A}_l} [\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+q} [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s_1-1)n+p, (r-1)n+q} \\ &\quad \times [\mathbf{R}_{\mathbf{W}_X}^{(l,p,q)}(z, \alpha)]_{(s-1)n+p, (s-1)n+q}. \end{aligned}$$

Similarly to inequality (49), we obtain the estimate

$$\begin{aligned} \max\{|[T_{11}^{(l,p,q)}]_{j,r,s}|, |[T_{12}^{(l,p,q)}]_{j,r,s}|, |[T_{13}^{(l,p,q)}]_{j,r,s}|, |[T_{14}^{(l,p,q)}]_{j,r,s}|\} \\ \leq \frac{C}{v^2} \sum_{r_1, s_1 \in \mathcal{A}_l} (|[\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+p}| + |[\mathbf{R}_{\mathbf{W}_X}(z, \alpha)]_{j, (r_1-1)n+q}|). \end{aligned} \quad (50)$$

Inequalities (50), (41), and (42) together imply that

$$|Q_3| \leq \frac{C\tau^2}{(nk)^2 v^5} \sum_{l=1}^m \sum_{p,q=1}^n \sigma_{pq}^{(l)} \leq \frac{C\tau^2}{v^5}. \quad (51)$$

3.3.4. Estimation of Q_4

Applying inequalities (50), (41), (42), and (46), we obtain the estimate

$$|Q_4| \leq \frac{C\tau^3}{v^6}. \quad (52)$$

3.3.5. Estimation of Q_5

We use inequalities (41), (42), and (46) again. We obtain

$$|Q_5| \leq \frac{C\tau}{v^4}. \quad (53)$$

3.3.6. Estimation of Q_6

Similarly to Q_5 , we apply inequalities (41), (42), and (45). We obtain

$$|Q_6| \leq \frac{C\tau}{v^4}. \quad (54)$$

Estimations (45), (49), and (51)–(54) together with the representation in (39) imply that

$$\left| \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_X(z, \alpha) - \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_Y \right| \leq \frac{C\tau}{v^4} + \frac{C\tau^3}{v^6} + \frac{C\tau^2}{v^5}. \quad (55)$$

Without loss of generality, we can assume that $\tau < 1$. The final estimate can be written as

$$\left| \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_X(z, \alpha) - \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_Y \right| \leq C\tau, \quad (56)$$

where the constant C depends on v , k , and m . Considering the transition to truncated values, the final estimate is as follows:

$$\left| \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_X(z, \alpha) - \frac{1}{nk} \mathbb{E} \operatorname{Tr} \mathbf{R}_Y \right| \leq C(\tau + (L_n(\tau))^{\frac{1}{2}}). \quad (57)$$

According to Remark 4, we can choose a sequence τ_n such that $\tau_n \rightarrow 0$ and $L_n(\tau_n) \rightarrow 0$ as $n \rightarrow \infty$. It follows that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\frac{1}{nk} \operatorname{Tr} \mathbf{R}_{W_X}(z, \alpha) - \frac{1}{nk} \operatorname{Tr} \mathbf{R}_{W_Y} \right) = 0. \quad (58)$$

To complete the proof, we give the following lemma.

3.4. Girko's Lemma

Lemma 2. Under conditions of Theorem 1, the following inequality holds:

$$\mathbb{E} \left| \frac{1}{nk} (\operatorname{Tr} \mathbf{R}_{W_X} - \mathbb{E} \operatorname{Tr} \mathbf{R}_{W_X}) \right|^2 \leq \frac{k^2}{nv^2}.$$

Proof. We prove this lemma using the method proposed by Girko (with a slight modification). For $l = 1, \dots, m$ and $j = 1, \dots, n$, we define σ -algebras $\mathfrak{M}^{(l,j)} = \sigma\{X_{pq}^{(s)}, s < l; p, q \leq j\}$. Note that $\mathfrak{M}^{(1,1)} = \{\emptyset, \Omega\}$ is a trivial σ -algebra and $\mathfrak{M}^{m,n}$ is a σ -algebra with respect to which all random variables $X_{pq}^{(l)}$ for $l = 1, m, 1 \leq p$, and $q \leq n$ are measurable. It is obvious that for any fixed $l = 1, \dots, m$, the σ -algebras of $\mathfrak{M}^{(l,j)}$ are non-decreasing for $j = 1, \dots, n$. Suppose that

$$\eta_{lj} = \frac{1}{nk} \left(\mathbb{E} \{ \operatorname{Tr} \mathbf{R}_{W_X}(z) | \mathfrak{M}^{(l,j)} \} - \mathbb{E} \{ \operatorname{Tr} \mathbf{R}_{W_X}(z) | \mathfrak{M}^{(l,j-1)} \} \right).$$

When $l = 1, \dots, m$ is fixed, the random variables are uncorrelated. It is easy to see that

$$\frac{1}{nk} \left(\operatorname{Tr} \mathbf{R}_{W_X}(z) - \mathbb{E} \operatorname{Tr} \mathbf{R}_{W_X}(z) \right) = \sum_{l=1}^m \sum_{j=1}^n \eta_{lj}.$$

The Cauchy inequality implies that

$$\mathbb{E} \left| \frac{1}{nk} \left(\text{Tr } \mathbf{R}_{W_X}(z) - \mathbb{E} \text{Tr } \mathbf{R}_{W_X}(z) \right) \right|^2 \leq \frac{m}{(nk)^2} \sum_{l=1}^m \mathbb{E} \left| \sum_{j=1}^n \eta_{lj} \right|^2 = \frac{m}{(nk)^2} \sum_{l=1}^m \sum_{j=1}^n \mathbb{E} |\eta_{lj}|^2. \quad (59)$$

For any fixed $l = 1, \dots, m$ and $j = 1, \dots, n$, define the matrix $\mathbf{W}^{(l,j)}$ obtained from the matrix $\mathbf{W}$ by removing, for any $p, q \in \mathcal{A}_l$, rows with numbers $(p-1)n+j$ and columns with numbers $(q-1)n+j$. Note that

$$\mathbb{E} \{ \text{Tr } \mathbf{R}_{W_X}^{(l,j)} | \mathfrak{M}^{(l,j)} \} - \mathbb{E} \{ \text{Tr } \mathbf{R}_{W_X} | \mathfrak{M}^{(l,j-1)} \} = 0.$$

Hence, we have the representation

$$\eta_{lj} = \frac{1}{nk} \left(\mathbb{E} \{ \text{Tr } \mathbf{R}_{W_X}(z) - \text{Tr } \mathbf{R}_{W_X}^{(l,j)} | \mathfrak{M}^{(l,j)} \} - \mathbb{E} \{ \text{Tr } \mathbf{R}_{W_X}(z) - \text{Tr } \mathbf{R}_{W_X}^{(l,j-1)} | \mathfrak{M}^{(l,j-1)} \} \right). \quad (60)$$

Now, we use the inequality

$$| \text{Tr } \mathbf{R}_{W_X}(z) - \text{Tr } \mathbf{R}_{W_X}^{(l,j)} | \leq \frac{|\mathcal{A}_l|}{v} \leq \frac{k^2}{v},$$

which, together with (59), gives the estimate

$$\mathbb{E} \left| \frac{1}{nk} \left(\text{Tr } \mathbf{R}_{W_X}(z) - \mathbb{E} \text{Tr } \mathbf{R}_{W_X}(z) \right) \right|^2 \leq \frac{k^2}{nv^2}.$$

The lemma has been proven. $\square$

Lemma 2, Remark 4, and relation (58) now complete the proof of the theorem.

4. Conclusions

Block-type random matrices play an important role in many studies. In this case, special attention is paid to the limiting distribution of the spectrum of such types of matrices. This is a difficult problem, but for special distributions of matrix elements, it is often possible to solve this problem. The universality of the limiting spectral distribution (its independence from the distribution of matrix elements) allows us to choose matrices with elements that have a distribution with certain good properties, for which special methods can be used, for example, for Gaussian distributions. However, this is not always possible. For example, even in the Gaussian case, there is no exact description of the limiting spectral distribution for the Teuplice matrix. It is known that it has an infinite carrier and is non-Gaussian (the second moment is 1 and the fourth moment is 8/3; see, for instance, [3]).

Many important problems in random matrix theory remain unsolved for block-type random matrices. First of all is the study of the distribution of eigennumbers in the local regime (convergence rate, rigidity, delocalisation of eigenvectors, and so on), and then the distribution of extreme values of eigennumbers, and so on.

Funding: This work was carried out within the framework of the project “Mathematical problems of the theory of stochastic and deterministic systems, including high-dimensional systems” (project no. 122040600066-5).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the author.

Conflicts of Interest: The author declares no conflict of interest.

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