

On Convergence of the Empirical Spectral Distribution Function for a Special Type of Random Matrices to the Normal Law*

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Abstract—Let $X_1, X_2, \dots, X_n$ be independent random variables, and let $\mathbf{E}^{(l)}$ for $l = 1, \dots, n$ be symmetric deterministic matrices. We study matrices of the type $\mathbf{W} = \sum_{l=1}^n X_l \mathbf{E}^{(l)}$. The convergence of empirical spectral distribution of $\mathbf{W}$ to the normal law is established under certain conditions on matrices $\mathbf{E}^{(l)}$. Applications include estimating the convergence rate for the spectral distribution functions of palindromic and circulant random matrices. The analysis employs a method introduced by the author in his 1980 work “On the Rate of Convergence in the Central Limit Theorem for Weakly Dependent Variables”.

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1. INTRODUCTION

We study the conditions under which symmetric random $n \times n$ matrices generated by n independent random variables $X_1, \dots, X_n$ have a normal limiting spectral distribution. We assume

$$\mathbb{E}X_j = 0, \quad \mathbb{E}X_j^2 = 1, \quad j = 1, \dots, n, \quad n \geq 1.$$

For any $n \geq 1$ and any $l = 1, \dots, n$ we consider symmetric $2n \times 2n$ matrices $\mathbf{E}^{(l)}$ with entries equal to 0 or 1 such that

$$\mathbf{E}^{(l)} \otimes \mathbf{E}^{(j)} = \mathbf{O} \quad \text{for } j \neq l, \quad 1 \leq j, l \leq n,$$

here and in what follows symbol $\otimes$ denotes Kronecker product of matrices and $\mathbf{O}$ denotes the zero matrix, and

$$\sum_{l=1}^n \mathbf{E}^{(l)} = \mathbf{1} \times \mathbf{1}^T, \tag{1}$$

where $\mathbf{1}$ denotes the column vector with all coordinates equal to 1.

We investigate the spectrum of matrices of the type

$$\mathbf{W} = \frac{1}{\sqrt{2n}} \sum_{l=1}^n X_l \mathbf{E}^{(l)}.$$

In such form we can represent different types of matrices like Töplitz matrices, Hankel matrices, circulant matrices and many other. We study the limit behavior of the distribution of the spectrum of random symmetric palindromic and circulant Töplitz matrices. A Töplitz matrix is called palindromic

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if its first row is a palindrome. Let $X_1, X_2, \dots, X_n$ be independent random variables with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = 1$. A palindromic symmetric Töplitz matrix has the form

$$\mathbf{P} = \frac{1}{\sqrt{2n}} \begin{bmatrix} X_1 & X_2 & X_3 \cdots & X_{n-1} & X_n & X_n \cdots & X_3 & X_2 & X_1 \\ X_2 & X_1 & X_2 \cdots & X_{n-2} & X_{n-1} & X_n \cdots & X_4 & X_3 & X_2 \\ X_3 & X_2 & X_1 \cdots & X_{n-3} & X_{n-2} & X_{n-1} \cdots & X_5 & X_4 & X_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ X_{n-1} & X_{n-2} & X_{n-3} \cdots & X_1 & X_2 & X_3 \cdots & X_n & X_{n-1} & X_{n-2} \\ X_n & X_{n-1} & X_{n-2} \cdots & X_2 & X_1 & X_2 \cdots & X_n & X_n & X_{n-1} \\ X_n & X_n & X_{n-1} \cdots & X_3 & X_2 & X_1 \cdots & X_{n-1} & X_n & X_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ X_4 & X_5 & X_6 \cdots & X_{n-1} & X_{n-2} & X_{n-3} \cdots & X_2 & X_3 & X_4 \\ X_3 & X_4 & X_5 \cdots & X_n & X_{n-1} & X_{n-2} \cdots & X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \cdots & X_n & X_n & X_{n-1} \cdots & X_2 & X_1 & X_2 \\ X_1 & X_2 & X_3 \cdots & X_{n-1} & X_n & X_n \cdots & X_3 & X_2 & X_1 \end{bmatrix}. \quad (2)$$

A circulant matrix is a Töplitz matrix whose rows are obtained from the first by a cyclic shift. The principal minor of a symmetric palindromic matrix obtained by deleting the last row and last column is a circulant matrix.

Let $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)^T$, $j = 1, \dots, 2n$, be vectors in $\mathbb{R}^{2n}$. We will use bold lowercase Latin letters to denote vectors, and bold uppercase Latin letters to denote matrices (random or deterministic).

For any $l = 1, \dots, 2n$ we define the matrices

$$\mathbf{E}^{(l)} := \begin{cases} \sum_{k=1}^{2n} \mathbf{e}_k \mathbf{e}_k^T + \mathbf{e}_1 \mathbf{e}_{2n}^T + \mathbf{e}_{2n} \mathbf{e}_1^T, & \text{for } l = 1, \\ \sum_{k=1}^{2n-l+1} (\mathbf{e}_k \mathbf{e}_{k+l-1}^T + \mathbf{e}_{k+l-1} \mathbf{e}_k^T) \\ + \sum_{k=2n-l+1}^{2n} (\mathbf{e}_k \mathbf{e}_{k-2n+l}^T + \mathbf{e}_{k-2n+l} \mathbf{e}_k^T) & \text{at } l = 2, \dots, n. \end{cases} \quad (3)$$

A palindromic symmetric Töplitz matrix generated by random variables $X_1, \dots, X_n$ can be written as

$$\mathbf{P} = \frac{1}{\sqrt{2n}} \sum_{l=1}^n X_l \mathbf{E}^{(l)}. \quad (4)$$

Along with palindromic Töplitz matrices, we will consider symmetric circulant matrices.

A circulant matrix of order n is formed by n elements of the first row, which we will denote by $X_1, \dots, X_n$. A circulant matrix is symmetric if $X_j = X_{n-j+2}$ for $j \geq 2$. A symmetric circulant matrix of odd order, say $2n - 1$, can be represented as

$$\mathbf{Q} = \frac{1}{\sqrt{2n-1}} \begin{bmatrix} X_1 & X_2 & X_3 \cdots & X_{n-1} & X_n & X_n \cdots & X_3 & X_2 \\ X_2 & X_1 & X_2 \cdots & X_{n-2} & X_{n-1} & X_n \cdots & X_4 & X_3 \\ X_3 & X_2 & X_1 \cdots & X_{n-3} & X_{n-2} & X_{n-1} \cdots & X_5 & X_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n-1} & X_{n-2} & X_{n-3} \cdots & X_1 & X_2 & X_3 \cdots & X_n & X_{n-1} \\ X_n & X_{n-1} & X_{n-2} \cdots & X_2 & X_1 & X_2 \cdots & X_n & X_n \\ X_n & X_n & X_{n-1} \cdots & X_3 & X_2 & X_1 \cdots & X_{n-1} & X_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_4 & X_5 & X_6 \cdots & X_{n-1} & X_{n-2} & X_{n-3} \cdots & X_2 & X_3 \\ X_3 & X_4 & X_5 \cdots & X_n & X_{n-1} & X_{n-2} \cdots & X_1 & X_2 \\ X_2 & X_3 & X_4 \cdots & X_n & X_n & X_{n-1} \cdots & X_2 & X_1 \end{bmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{2n}$ be the eigenvalues of the matrix $\mathbf{W}$. Let us define the empirical spectral distribution function

$$F_n(x) = \frac{1}{2n} \sum_{j=1}^{2n} \mathbb{I}\{\lambda_j < x\},$$

where $\mathbb{I}\{A\}$ denotes the indicator of event A .

In [4], it was shown that, for Töplitz matrices, there exists a limit spectral distribution with infinite support and different from the normal one (the second moment is 1, the fourth $8/3$). In [2], symmetric palindromic Töplitz matrices were considered and it was shown that their limit spectral distribution is normal. In both papers, the method of moments was used and a sufficiently large number of moments were required. In [7], symmetric circulant block-matrices with large random blocks was considered.

For any matrix $\mathbf{A}$ we denote by symbol $\|\mathbf{A}\|$ the operator norm of matrix $\mathbf{A}$ and by symbol $\|\mathbf{A}\|_2$ we denote the Frobenius norm. We prove the following result.

Theorem 1. *Let matrix $\mathbf{W}$ has the form (4) with some matrices $\mathbf{E}^{(l)}$. Suppose that there exist a constant C_1 and a sequence β_{n1} such that for any $n \geq 1$ and any $l = 1, \dots, n$*

$$\|\mathbf{E}_l\| \leq C_1 \beta_{n1};$$

and there exists a constant C_2 such that

$$\left\| \frac{1}{2n} \sum_{l=1}^n \mathbf{E}_l^2 - \mathbf{I} \right\|_2 \leq C_2 \beta_{n2}. \quad (5)$$

Suppose as well that

$$\sup_{j \geq 1} \mathbb{E}|X_j|^3 \leq \mu_3.$$

Then

$$\sup_x |\mathbb{E}F_n(x) - \Phi(x)| \leq C \left(\frac{\sqrt{\beta_{n2}} \log n}{\sqrt{n}} + \frac{\beta_{n1}^{\frac{3}{2}}}{n^{\frac{1}{4}}} + \frac{\beta_{n1}^2}{n^{\frac{3}{4}}} \right). \quad (6)$$

Assuming that

$$\mathbb{E}|X_j|^4 \leq \mu_4 < \infty,$$

we have

$$\mathbb{E} \sup_x |F_n(x) - \Phi(x)| \leq C \mu_3 \left(\frac{\sqrt{\beta_{n2}} \log n}{\sqrt{n}} + \frac{\beta_{n1}}{n^{\frac{1}{8}}} + \frac{\beta_{n1}^2}{n^{\frac{1}{4}}} \right) + \frac{C \sqrt{\mu_4} \beta_{n2}^2}{n^{\frac{1}{4}}} + \frac{\beta_{n1}^3 \mu_3}{n^{\frac{3}{4}}}. \quad (7)$$

Proposition 1. *Let the following conditions be met:*

$$\lim_{n \rightarrow \infty} \frac{\beta_{n1}^3}{\sqrt{n}} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{\beta_{n2}}{\sqrt{n}} = 0.$$

Then

$$\lim_{n \rightarrow \infty} \sup_x |F_n(x) - \Phi(x)| = 0 \quad \text{in probability.}$$

We give the proof of Proposition 1 in Section 2 after the proof of Theorem 1.

Let $F_{\mathbf{P}}(x)$ and $F_{\mathbf{Q}}(x)$ denote empirical spectral distributions of matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively. As a consequence of Theorem 1 we obtain results for palindromic and circulant matrices. It is known that by virtue of the "rank" inequality, see [1, Theorem A.43], the inequality

$$\sup_x |F_{\mathbf{P}}(x) - F_{\mathbf{Q}}(x)| \leq \frac{1}{2n}. \quad (8)$$

Theorem 2. *Let there exists a constant $0 < \mu_3 < \infty$ such that*

$$\sup_{j \geq 1} \mathbb{E}|X_j|^3 \leq \mu_3.$$

Then there exists a constant $C > 0$ such that

$$\sup_x |\mathbb{E}F_{\mathbf{P}}(x) - \Phi(x)| \leq \frac{C \mu_3}{n^{\frac{1}{4}}}. \quad (9)$$

The theorem and inequality (8) immediately imply the following corollary.

Corollary 1. *Let there exists a constant $0 < \mu_3 < \infty$ such that*

$$\sup_{j \geq 1} \mathbb{E}|X_j|^3 \leq \mu_3.$$

Then there exists a constant $C > 0$ such that

$$\sup_x |\mathbb{E}F_{\mathbf{Q}}(x) - \Phi(x)| \leq \frac{C \mu_3}{n^{\frac{1}{4}}}. \quad (10)$$

We prove as well the result concerning empirical spectral distribution function.

Theorem 3. *Let there exists a constant $0 < \mu_3 < \infty$ such that*

$$\sup_{j \geq 1} \mathbb{E}|X_j|^3 \leq \mu_3 \quad \text{and} \quad \sup_{j \geq 1} \mathbb{E}|X_j|^4 \leq \mu_4. \quad (11)$$

Then there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_x |F_{\mathbf{P}}(x) - \Phi(x)| \leq \frac{C \mu_3}{n^{\frac{1}{8}}} + \frac{C \sqrt{\mu_4}}{n^{\frac{1}{4}}}.$$

The theorem and inequality (8) immediately imply the following corollary.

Corollary 2. *Let there exists a constant $0 < \mu_3 < \infty$ such that*

$$\sup_{j \geq 1} \mathbb{E}|X_j|^3 \leq \mu_3 \quad \text{and} \quad \sup_{j \geq 1} \mathbb{E}|X_j|^4 \leq \mu_4.$$

Then there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_x |F_{\mathbf{Q}}(x) - \Phi(x)| \leq \frac{C\mu_3}{n^{\frac{1}{8}}} + \frac{C\sqrt{\mu_4}}{n^{\frac{1}{4}}}.$$

In Section 4 we improve inequalities (9) and 10. We prove the following

Theorem 4. *For the circulant matrix $\mathbf{Q}$ the following estimate is holds:*

$$\sup_x |\mathbb{E} F_{\mathbf{Q}}(x) - \Phi(x)| \leq \frac{C\mu_3}{\sqrt{n}}, \quad (12)$$

and for palindromic matrices $\mathbf{P}$,

$$\sup_x |\mathbb{E} F_{\mathbf{P}}(x) - \Phi(x)| \leq \frac{C\mu_3}{\sqrt{n}}. \quad (13)$$

2. PROOF OF THEOREM 1

To prove inequality (6), we use the method proposed by the author in [9]. To prove inequality (7), we estimate the variance of the characteristic function of the empirical spectral distribution function of the matrix $\mathbf{W}$.

First, consider the characteristic function of the expected empirical spectral distribution function

$$f_n(t) = \frac{1}{2n} \sum_{j=1}^{2n} \mathbb{E} \exp\{it\lambda_j\} = \frac{1}{2n} \text{Tr} \mathbb{E} \exp\{it\mathbf{W}\}.$$

We will study the behavior of the derivative of the characteristic function $f'_n(t)$,

$$f'_n(t) = \frac{i}{2n} \mathbb{E} \text{Tr} \mathbf{W} \exp\{it\mathbf{W}\} = \frac{i}{2n\sqrt{2n}} \sum_{j=1}^n \mathbb{E} X_l \text{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}.$$

Introduce the matrices $\mathbf{W}^{(l)} = \frac{1}{\sqrt{2n}} \sum_{k \neq l} \mathbb{E} X_k \mathbf{E}^{(k)}$. Using the independence of the random variables X_l and the matrices $\mathbf{W}^{(l)}$, $l = 1, \dots, n$, we arrive at the equality

$$f'_n(t) = \frac{i}{2n\sqrt{2n}} \sum_{l=1}^n \mathbb{E} X_l \text{Tr} \left(\exp\{it\mathbf{W}\} - \exp\{it\mathbf{W}^{(l)}\} \right) \mathbf{E}^{(l)}.$$

Let us now use the Duhamel formula: for any symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, the following equality holds:

$$\exp\{it(\mathbf{A} + \mathbf{B})\} = \exp\{it\mathbf{A}\} + i \int_0^t \exp\{is\mathbf{A}\} \mathbf{B} \exp\{i(t-s)(\mathbf{A} + \mathbf{B})\} ds.$$

See, for example, [8, Formula (2.14)]. We obtain

$$f'_n(t) = \frac{i}{(2n)^2} \sum_{l=1}^n \int_0^t \mathbb{E} X_l^2 \text{Tr} \exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s)\mathbf{W}\} \mathbf{E}^{(l)} ds.$$

Applying the Duhamel formula once more and using the independence of the random variables X_l and the matrices $\mathbf{W}^{(l)}$, $l = 1, \dots, n$, we can write

$$f'_n(t) = G_1(t) + G_2(t), \quad (14)$$

where

$$G_1(t) = \frac{i}{(2n)^2} \sum_{l=1}^n \int_0^t \mathbb{E} \operatorname{Tr} \exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds,$$

$$G_2(t) = \frac{i}{(2n)^2 \sqrt{2n}} \sum_{l=1}^n \mathbb{E} X_l^3 \int_0^t ds \left[\operatorname{Tr} \exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \right. \\ \left. \times \int_0^{t-s} \exp\{is_1\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s-s_1)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds_1 \right].$$

Let us first estimate the quantity $G_2(t)$.

Lemma 1. *Under the conditions of the theorem, there exists a constant $C > 0$ such that*

$$|G_2(t)| \leq C \mu_3 \frac{t^2 \beta_{n1}^3(t)}{\sqrt{n}}.$$

Proof. Since $\exp\{is\mathbf{A}\}$ is a unitary matrix, its operator norm is 1. Since the operator norm is submultiplicative, the following estimate holds:

$$|\exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{is_1\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s-s_1)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)}| \leq \beta_{n1}^3(t).$$

This implies that

$$\left| \operatorname{Tr} \exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \int_0^{t-s} \exp\{is_1\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s-s_1)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds_1 \right| \\ \leq n \beta_{n1}^3(t) |t-s|.$$

In turn, the last inequality leads to the estimate

$$|G_2(t)| \leq \frac{C t^2 \mu_3 \beta_{n1}^3(t)}{\sqrt{n}}.$$

The lemma is proved. □

Continuing equality (14), we can write

$$G_1(t) = G_{11}(t) + G_{12}(t) + G_{13}(t),$$

where

$$G_{11}(t) = \frac{i}{(2n)^2} \sum_{l=1}^n \int_0^t \mathbb{E} \operatorname{Tr} \exp\{is\mathbf{W}\} \mathbf{E}^{(l)} \exp\{i(t-s)\mathbf{W}\} \mathbf{E}^{(l)} ds,$$

$$G_{12}(t) = -\frac{i}{(2n)^2 \sqrt{2n}} \sum_{l=1}^n \mathbb{E} X_l \int_0^t \int_0^s \operatorname{Tr} \exp\{is_1\mathbf{W}\} \mathbf{E}^{(l)} \exp\{i(s-s_1)\mathbf{W}^{(l)}\} \\ \times \mathbf{E}^{(l)} \exp\{i(t-s)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds_1 ds,$$

$$G_{13}(t) = -\frac{i}{(2n)^2 \sqrt{2n}} \sum_{l=1}^n \mathbb{E} X_l \int_0^t \int_0^{t-s} \operatorname{Tr} \exp\{is\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{is_1\mathbf{W}\} \\ \times \mathbf{E}^{(l)} \exp\{i(t-s-s_1)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds ds_1.$$

Lemma 2. *Under the conditions of the theorem, there exists a constant C such that*

$$\max |G_{12}(t)|, |G_{13}(t)| \leq C \mu_3 \frac{t^2 \beta_{n1}^3(t)}{\sqrt{n}}.$$

Proof. The proof repeats that of Lemma 1. To complete the estimate, one must use the corollary of Lyapunov's inequality for moments,

$$\mathbb{E} X_l^2 \mathbb{E} |X_l| \leq \mathbb{E} |X_l|^3. \quad \square$$

It remains for us to find the asymptotics of $G_{11}(t)$. Let $\mathbf{U} = \mathbf{U}(t) = \exp\{it\mathbf{W}\}$. We represent $G_{11}(t)$ in the form

$$G_{11}(t) = \frac{-1}{(2n)^2} \sum_{l=1}^n \int_0^t \mathbb{E} \operatorname{Tr} \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(t-s) \mathbf{E}^{(l)} ds.$$

Further, let $[\mathbf{A}, \mathbf{B}] := \mathbf{AB} - \mathbf{BA}$ denote the commutator of matrices $\mathbf{A}$ and $\mathbf{B}$. Let us prove the following lemma.

Lemma 3. *The following relation is valid:*

$$\begin{aligned} \mathbb{E} \operatorname{Tr} \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(t-s) \mathbf{E}^{(l)} \\ = \operatorname{Tr} \mathbf{U}(t) (\mathbf{E}^{(l)})^2 + i \int_0^s \operatorname{Tr} \mathbf{U}(r) [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(-r) \mathbf{U}(t) \mathbf{E}^{(l)} dr. \end{aligned}$$

Proof. Introduce the matrix function

$$\mathbf{K}_l(s) := \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(-s).$$

It is easy to see that

$$\mathbf{K}_l'(s) = i \left(\mathbf{W} \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(-s) - \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{W} \mathbf{U}(-s) \right) = i \mathbf{U}(s) [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(-s).$$

We used the fact that

$$\frac{d}{ds} \mathbf{U}(\pm s) = \pm i \mathbf{W} \mathbf{U}(\pm s) = \pm i \mathbf{U}(\pm s) \mathbf{W}.$$

Since $\mathbf{K}_l(0) = \mathbf{E}^{(l)}$, we can write

$$\begin{aligned} \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(t-s) \mathbf{E}^{(l)} &= \mathbf{U}(s) \mathbf{E}^{(l)} \mathbf{U}(-s) \mathbf{U}(t) \mathbf{E}^{(l)} \\ &= \mathbf{E}^{(l)} \mathbf{U}(t) \mathbf{E}^{(l)} + i \int_0^s \mathbf{U}(r) [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(-r) \mathbf{U}(t) \mathbf{E}^{(l)} dr. \end{aligned}$$

Taking the trace and the mathematical expectation of both sides of the last equality and considering that $\operatorname{Tr} \mathbf{E}^{(l)} \mathbf{U}(t) \mathbf{E}^{(l)} = \operatorname{Tr} \mathbf{U}(t) (\mathbf{E}^{(l)})^2$, we obtain the required result. The lemma is proved. $\square$

Applying Lemma 3 to the function $G_{11}(t)$, we obtain

$$G_{11}(t) = -t \frac{1}{(2n)^2} \sum_{l=1}^n \operatorname{Tr} \mathbf{U}(t) (\mathbf{E}^{(l)})^2 + R(t),$$

where

$$R(t) = -\frac{i}{(2n)^2} \sum_{l=1}^n \int_0^t \int_0^s \mathbb{E} \operatorname{Tr} \mathbf{U}(r) \mathbf{E}^{(l)} [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)} dr ds.$$

Condition (5) and the last relation together imply

$$f_n'(t) = -t f_n(t) + \frac{\theta(t) C_2 \beta_{n2} |t|}{\sqrt{n}} + \frac{C \mu_3 \beta_{n1}^3 t^2}{\sqrt{n}} + R(t), \quad (15)$$

where $|\theta(t)| \leq 1$. We use here that, for any matrices $\mathbf{A}$ and $\mathbf{B}$, $|\operatorname{Tr} \mathbf{AB}| \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$ and that unitary matrix $\mathbf{U}$ has Frobenius norm equal to $\sqrt{2n}$.

It remains to estimate $R(t)$.

Lemma 4. *Under the conditions of the theorem, the following inequality holds:*

$$\left| \mathbb{E} \operatorname{Tr} \mathbf{U}(r) \mathbf{E}^{(l)} [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)} \right| \leq C \mu_3 \beta_{n1}^5 \max\{r, t-r\}.$$

Proof. Using the representation for $\mathbf{W}$, we can write

$$\begin{aligned} \mathbb{E} \operatorname{Tr} \mathbf{U}(r) \mathbf{E}^{(l)} [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)} \\ = \frac{1}{\sqrt{2n}} \sum_{k=1}^n \mathbb{E} X_k \operatorname{Tr} \mathbf{U}(r) \mathbf{E}^{(l)} [\mathbf{E}^{(k)}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)}. \end{aligned}$$

Next, we use the Duhamel formula

$$\exp\{is\mathbf{W}\} = \exp\{is\mathbf{W}^{(k)}\} + iX_k \int_0^s \exp\{ir\mathbf{W}^{(k)}\} \mathbf{E}^{(k)} \exp\{i(s-r)\mathbf{W}\} dr.$$

We arrive at the following representation:

$$\begin{aligned} \mathbb{E} \operatorname{Tr} \mathbf{U}(r) \mathbf{E}^{(l)} [\mathbf{W}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)} \\ = \frac{1}{\sqrt{2n}} \sum_{k=1}^n \mathbb{E} X_k \operatorname{Tr} \mathbf{U}^{(k)}(r) \mathbf{E}^{(l)} [\mathbf{E}^{(k)}, \mathbf{E}^{(l)}] \mathbf{U}^{(k)}(t-r) \mathbf{E}^{(l)} \\ + r_1(t, s) + r_2(t, s), \end{aligned}$$

where

$$\begin{aligned} \mathbf{U}^{(k)}(s) &= \exp\{is\mathbf{W}^{(k)}\}, \\ r_1(t, s) &= i \frac{1}{2n} \sum_{k=1}^n \mathbb{E} X_k^2 \operatorname{Tr} \int_0^s \mathbf{U}^{(k)}(r) \mathbf{E}^{(k)} \mathbf{U}(s-r) dr \mathbf{E}^{(l)} [\mathbf{E}^{(k)}, \mathbf{E}^{(l)}] \mathbf{U}(t-r) \mathbf{E}^{(l)}, \\ r_2(t, s) &= i \frac{1}{2n} \sum_{k=1}^n \mathbb{E} X_k^2 \operatorname{Tr} \int_0^{t-s} \mathbf{U}^{(k)}(s) \mathbf{E}^{(l)} [\mathbf{E}^{(k)}, \mathbf{E}^{(l)}] \mathbf{U}^{(k)}(r) \mathbf{E}^{(k)} \mathbf{U}(t-s-r) dr \mathbf{E}^{(l)}. \end{aligned}$$

Due to the independence of the random variables X_k and the matrices $\mathbf{U}^{(k)}(r)$,

$$\frac{1}{\sqrt{2n}} \sum_{k=1}^n \mathbb{E} X_k \operatorname{Tr} \mathbf{U}^{(k)}(r) \mathbf{E}^{(l)} [\mathbf{E}^{(k)}, \mathbf{E}^{(l)}] \mathbf{U}^{(k)}(t-r) \mathbf{E}^{(l)} = 0.$$

$$\begin{aligned} |r_1(t, s)| &\leq C\mu_3\beta_{n1}^5|s|, \\ |r_2(t, s)| &\leq C\mu_3\beta_{n1}^5|t-s|. \end{aligned}$$

The lemma is proved. $\square$

Lemma 4 implies the estimate

$$|R(t)| \leq \frac{C\mu_3|t|^3\beta_{n1}^5}{n}.$$

From this and from representation (15), it follows that

$$f'_n(t) = -tf_n(t) + C\theta(t)\mu_3 \left(\frac{|t|\beta_{n2} + t^2\beta_{n1}^3}{\sqrt{n}} + \frac{|t|^3\beta_{n1}^5}{n} \right). \quad (16)$$

Here $\theta(t)$ denotes a function such that $|\theta(t)| \leq 1$. Integrating the last equality with $f_n(0) = 1$, we obtain

$$f_n(t) = \exp\{-t^2/2\} + C\mu_3\theta(t) \left(\frac{\min(|t|, 1)\beta_{n2} + |t|\beta_{n1}^3}{\sqrt{n}} + \frac{|t|^2\beta_{n1}^5}{n} \right).$$

The obtained representation and Esseen's inequality (see [3, Theorem 2.1]) imply the required estimate. See details in [9]. Thus, inequality (6) is proved.

As already mentioned, to prove inequality (7) we estimate the variance of the characteristic function of the empirical spectral measure of the matrix $\mathbf{W}$.

Lemma 5. *Under the conditions of Theorem 1, there exists a constant $C > 0$ such that*

$$\mathbb{E} \left| \frac{1}{n} \left(\text{Tr} \exp\{it\mathbf{W}\} - \mathbb{E} \text{Tr} \exp\{it\mathbf{W}\} \right) \right|^2 \leq \frac{Ct^2 \sqrt{\beta_{n1}}}{\sqrt{n}} + \frac{C\mu_4 t^4 \beta_{n1}^4}{n}.$$

Proof. To prove the lemma, we use the martingale representation of the trace. This technique was apparently first applied in random matrix theory by V. Girko. Consider the increasing sequence of sigma-algebras $\mathfrak{M}_l = \sigma(X_j, 1 \leq j \leq l)$, $l = 1, \dots, n$. For $l = 1, \dots, n$, consider the quantities

$$\gamma_l = \mathbb{E} \left[\frac{1}{n} \text{Tr} \exp\{it\mathbf{W}\} \middle| \mathfrak{M}_l \right] - \mathbb{E} \left[\frac{1}{n} \text{Tr} \exp\{it\mathbf{W}\} \middle| \mathfrak{M}_{l-1} \right].$$

Here and thereafter, by $\mathfrak{M}_0$ we understand the trivial sigma-algebra, so that $\mathbb{E} \left[\frac{1}{n} \text{Tr} \exp\{it\mathbf{W}\} \middle| \mathfrak{M}_0 \right] = \mathbb{E} \left[\frac{1}{n} \text{Tr} \exp\{it\mathbf{W}\} \right]$. In the introduced notation, we can write

$$\mathbb{E} \left| \frac{1}{n} \left(\text{Tr} \exp\{it\mathbf{W}\} - \mathbb{E} \text{Tr} \exp\{it\mathbf{W}\} \right) \right|^2 = \mathbb{E} \left| \sum_{l=1}^n \gamma_l \right|^2 = \sum_{l=1}^n \mathbb{E} |\gamma_l|^2.$$

Next, note that

$$\mathbb{E} \left[\frac{1}{2n} \text{Tr} \exp\{it\mathbf{W}^{(l)}\} \middle| \mathfrak{M}_l \right] = \mathbb{E} \left[\frac{1}{n} \text{Tr} \exp\{it\mathbf{W}^{(l)}\} \middle| \mathfrak{M}_{l-1} \right].$$

From this it follows that

$$\gamma_l = \hat{\gamma}_l - \tilde{\gamma}_l,$$

where

$$\begin{aligned} \hat{\gamma}_l &= \mathbb{E} \left[\frac{1}{2n} (\text{Tr} \exp\{it\mathbf{W}\} - \text{Tr} \exp\{it\mathbf{W}^{(l)}\}) \middle| \mathfrak{M}_l \right], \\ \tilde{\gamma}_l &= \mathbb{E} \left[\frac{1}{2n} (\text{Tr} \exp\{it\mathbf{W}\} - \text{Tr} \exp\{it\mathbf{W}^{(l)}\}) \middle| \mathfrak{M}_{l-1} \right]. \end{aligned}$$

Since

$$\mathbb{E} |\gamma_l|^2 \leq 2(\mathbb{E} |\tilde{\gamma}_l|^2 + \mathbb{E} |\hat{\gamma}_l|^2),$$

and, in turn,

$$\max\{\mathbb{E} |\tilde{\gamma}_l|^2, \mathbb{E} |\hat{\gamma}_l|^2\} \leq \mathbb{E} \left| \frac{1}{n} \left(\text{Tr} \exp\{it\mathbf{W}\} - \text{Tr} \exp\{it\mathbf{W}^{(l)}\} \right) \right|^2,$$

we have

$$\begin{aligned} \mathbb{E} \left| \frac{1}{2n} \left(\text{Tr} \exp\{it\mathbf{W}\} - \mathbb{E} \text{Tr} \exp\{it\mathbf{W}\} \right) \right|^2 \\ \leq \frac{4}{n^2} \sum_{l=1}^n \mathbb{E} \left| \text{Tr} \left(\exp\{it\mathbf{W}\} - \exp\{it\mathbf{W}^{(l)}\} \right) \right|^2. \end{aligned}$$

Furthermore, applying Duhamel formula, we get

$$\text{Tr} \exp\{it\mathbf{W}\} - \text{Tr} \exp\{it\mathbf{W}^{(l)}\} = Q_1 + Q_2,$$

where

$$\begin{aligned} Q_1^{(l)} &= \frac{it}{\sqrt{2n}} X_l \text{Tr} \exp\{it\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \\ Q_2^{(l)} &= -\frac{1}{2n} X_l^2 \int_0^t \int_0^{t-s} \text{Tr} \exp\{it\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \\ &\quad \times \exp\{i(t-s)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} \exp\{i(t-s-r)\mathbf{W}\} \mathbf{E}^{(l)} dr ds. \end{aligned}$$

It is not difficult to obtain that

$$\frac{1}{(2n)^2} \sum_{l=1}^n \mathbb{E} |Q_2^{(l)}|^2 \leq C\mu_4 |t|^4 \beta_{n1}^3 / n.$$

For the estimation of $\frac{1}{(2n)^2} \mathbb{E} |Q_1^{(l)}|^2$, we note that

$$\mathbb{E} |Q_1^{(l)}|^2 = \frac{t^2}{2n} \mathbb{E} |\operatorname{Tr} \exp\{it\mathbf{W}^{(l)}\} \mathbf{E}^{(l)}|^2.$$

Applying Duhamel formula, we get

$$\begin{aligned} \mathbb{E} |Q_1^{(l)}|^2 &\leq \frac{Ct^2}{2n} \mathbb{E} |\operatorname{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}|^2 \\ &\quad + \frac{Ct^2}{n^2} \mathbb{E} X_l^2 \left| \int_0^t \operatorname{Tr} \exp\{is\mathbf{W}\} \mathbf{E}^{(l)} \exp\{i(t-s)\mathbf{W}^{(l)}\} \mathbf{E}^{(l)} ds \right|^2. \end{aligned}$$

Continuing this inequality, we obtain

$$\frac{1}{(2n)^2} \sum_{l=1}^n \mathbb{E} |Q_1^{(l)}|^2 \leq \frac{Ct^2}{(2n)^3} \sum_{l=1}^n \mathbb{E} |\operatorname{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}|^2 + \frac{C|t|^4 \beta_{n1}^4}{n}. \quad (17)$$

It remains to estimate the first term on the right-hand side of (17). Recall notation $\mathbf{U} = \exp\{it\mathbf{W}\}$. We can write

$$\sum_{l=1}^n |\operatorname{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}|^2 \leq \sum_{l=1}^n \sum_{j,k=1}^{2n} \sum_{p,q=1}^{2n} |[\mathbf{U}]_{jk}| |[\mathbf{U}]_{pq}| \mathbf{E}_{kj}^{(l)} \mathbf{E}_{pq}^{(l)}.$$

Since for any $j, k = 1, \dots, 2n$, $\sum_{l=1}^n \mathbf{E}_{jk}^{(l)} = 1$ holds and $\mathbb{E}_{jk}^{(l)} = 0$ or 1 , there exists only one number $l_0 = l(j, k)$ such that $\mathbf{E}_{jk}^{(l)} = 0$ for $l \neq l_0$. This implies that

$$\sum_{l=1}^n |\operatorname{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}|^2 \leq \sum_{j,k=1}^{2n} |[\mathbf{U}]_{jk}| \sum_{p,q=1}^{2n} |[\mathbf{U}]_{pq}| \mathbf{E}_{pq}^{(l_0)}.$$

Furthermore, applying Cauchy's inequality, we get, for any $j, k = 1, \dots, 2n$,

$$\sum_{p,q=1}^{2n} |[\mathbf{U}]_{pq}| \mathbf{E}_{pq}^{(l_0)} \leq Cn \sqrt{\beta_{n1}}.$$

From this it follows that

$$\frac{1}{(2n)^3} \sum_{l=1}^n |\operatorname{Tr} \exp\{it\mathbf{W}\} \mathbf{E}^{(l)}|^2 \leq \frac{Ct^2 \sqrt{\beta_{n1}}}{\sqrt{n}}$$

and

$$\mathbb{E} \left| \frac{1}{2n} \left(\operatorname{Tr} \exp\{it\mathbf{W}\} - \mathbb{E} \operatorname{Tr} \exp\{it\mathbf{W}\} \right) \right|^2 \leq \frac{Ct^2 \sqrt{\beta_{n1}}}{\sqrt{n}} + \frac{C\mu_4 t^4 \beta_{n1}^4}{n}. \quad (18)$$

Thus, the lemma is proved. $\square$

Let $\varphi(t) = \exp\{-t^2/2\}$. To prove Theorem 1, it now suffices to note that

$$\mathbb{E} \left| \frac{1}{2n} \operatorname{Tr} \mathbf{U}(t) - \varphi(t) \right| \leq \left| \varphi(t) - \mathbb{E} \frac{1}{n} \operatorname{Tr} \mathbf{U}(t) \right| + \mathbb{E} \left| \frac{1}{n} \operatorname{Tr} \mathbf{U}(t) - \mathbb{E} \frac{1}{n} \operatorname{Tr} \mathbf{U}(t) \right|.$$

Representation (16) and inequality (18) together imply that

$$\mathbb{E} \left| \frac{1}{2n} \operatorname{Tr} \mathbf{U}(t) - \varphi(t) \right| \leq \frac{C\mu_3 |t| \beta_{n1}^{\frac{1}{4}}}{n^{\frac{1}{4}}}$$

$$+ C\mu_3 \left(\frac{\min(|t|, 1)\beta_{n2} + |t|\beta_{n1}^3}{\sqrt{n}} + \frac{|t|^2\beta_{n1}^5}{n} \right) + \frac{C\mu_4^{\frac{1}{2}}t^2\beta_{n1}^2}{\sqrt{n}}. \quad (19)$$

Applying now Esseen's inequality for the region $|t| \leq Cn^{\frac{1}{8}}/\beta_{n1}$ (see [3, Theorem 2.1]), we obtain the required result. Thus, Theorem 1 is proved.

To prove Proposition 1, we note that inequality (19) imply that, under conditions of Proposition 1,

$$f_n(t) = \frac{1}{2n} \sum_{j=1}^{2n} \exp\{it\lambda_j\} \rightarrow \exp\left\{-\frac{t^2}{2}\right\} \quad \text{in probability as } n \rightarrow \infty.$$

This implies the result of Proposition 1.

3. APPLICATION OF THE MAIN RESULT TO THE PALINDROMIC MATRICES. PROOF OF THEOREM 2

In this Section we consider application of the Theorem 1 to the palindromic matrices. As noted in the representation (2), the matrices $\mathbf{P}$ have the form (1) with the matrices $\mathbf{E}^{(l)}$ given by equality (3). This equality is of the form

$$\mathbf{E}^{(l)} := \begin{cases} \sum_{k=1}^{2n} \mathbf{e}_k \mathbf{e}_k^T + \mathbf{e}_1 \mathbf{e}_{2n}^T + \mathbf{e}_{2n} \mathbf{e}_1^T & \text{for } l = 1, \\ \sum_{k=1}^{2n-l+1} (\mathbf{e}_k \mathbf{e}_{k+l-1}^T + \mathbf{e}_{k+l-1} \mathbf{e}_k^T) \\ \quad + \sum_{k=2n-l+1}^{2n} (\mathbf{e}_k \mathbf{e}_{k-2n+l}^T + \mathbf{e}_{k-2n+l} \mathbf{e}_k^T) & \text{at } l = 2, \dots, n, \end{cases} \quad (20)$$

where $\mathbf{e}_j$ denote unit vectors in $\mathbb{R}^{2n}$, $j = 1, \dots, 2n$.

Proof of Theorem 2. To prove Theorem 2, it suffices to estimate the values β_{n1} and β_{n2} . It is obvious that the matrix $\mathbf{E}^{(l)}$, $l = 1, \dots, n$, contains no more than 4 nonzero elements equal to 1 in each row. Consequently, its spectral norm does not exceed 4. The following estimate holds:

$$\beta_{n1} \leq 4.$$

To estimate β_{n2} , we will prove the following lemma.

Lemma 6. *For the matrices $\mathbf{E}^{(l)}$, $l = 1, \dots, n$, defined in (20), the following equality holds:*

$$\sum_{l=1}^n (\mathbf{E}^{(l)})^2 = \mathbf{B} = \begin{bmatrix} 2n-1 & 1 & 1 & \cdots & 1 & 1 & 2n \\ 1 & 2n-1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2n & 1 & 1 & \cdots & 1 & 1 & 2n-1 \end{bmatrix}.$$

Proof. First, consider $l = 1$. It is easy to see that

$$(\mathbf{E}^{(1)})^2 = \mathbf{I} + 2\mathbf{e}_1 \mathbf{e}_{2n}^T + 2\mathbf{e}_{2n} \mathbf{e}_1^T.$$

Further, for $l \geq 2$ we can write

$$\begin{aligned} (\mathbf{E}^{(l)})^2 &= \sum_{j=1}^{2n-l+1} \sum_{k=1}^{2n-l+1} (\mathbf{e}_j \mathbf{e}_{j+l-1}^T + \mathbf{e}_{j+l-1} \mathbf{e}_j^T) (\mathbf{e}_k \mathbf{e}_{k+l-1}^T + \mathbf{e}_{k+l-1} \mathbf{e}_k^T) \\ &\quad + \sum_{j=2n-l+2}^{2n} \sum_{k=2n-l+2}^{2n} (\mathbf{e}_j \mathbf{e}_{j+l-2n}^T + \mathbf{e}_{j+l-2n} \mathbf{e}_j^T) (\mathbf{e}_k \mathbf{e}_{k+l-2n}^T + \mathbf{e}_{k+l-2n} \mathbf{e}_k^T) \\ &\quad + 2 \sum_{j=1}^{2n-l+1} \sum_{k=2n-l+2}^{2n} (\mathbf{e}_j \mathbf{e}_{j+l-1}^T + \mathbf{e}_{j+l-1} \mathbf{e}_j^T) (\mathbf{e}_k \mathbf{e}_{k+l-2n}^T + \mathbf{e}_{k+l-2n} \mathbf{e}_k^T). \end{aligned}$$

Considering the orthogonality of the vectors $\mathbf{e}_j, j = 1, \dots, 2n$, we arrive at the following equality

$$\begin{aligned} (\mathbf{E}^{(l)})^2 &= \sum_{j=1}^{2n-l+1} (\mathbf{e}_{j+l-1} \mathbf{e}_{j+l-1}^T + \mathbf{e}_j \mathbf{e}_j^T) + \sum_{j=1}^{2n-l+1} (\mathbf{e}_j \mathbf{e}_{j+2l-2}^T + \mathbf{e}_{j+2l-2} \mathbf{e}_j^T) \\ &+ \sum_{j=2n-l+2}^{2n} (\mathbf{e}_j \mathbf{e}_j^T + \mathbf{e}_{j+l-2n} \mathbf{e}_{j+l-2n}^T) + 2\mathbf{e}_{2n} \mathbf{e}_1^T + 2\mathbf{e}_1 \mathbf{e}_{2n}^T \\ &= 2 \sum_{j=1}^{2n} \mathbf{e}_j \mathbf{e}_j^T + 2\mathbf{e}_{2n} \mathbf{e}_1^T + 2\mathbf{e}_1 \mathbf{e}_{2n}^T \\ &+ \sum_{j=1}^{2n-l+1} (\mathbf{e}_j \mathbf{e}_{j+2l-2}^T + \mathbf{e}_{j+2l-2} \mathbf{e}_j^T). \end{aligned}$$

Summing over $l = 1, \dots, n$, we get the required result. $\square$

Lemma 6 implies that

$$\beta_{n2} \leq 2.$$

Applying now the result of Theorem 1, we get

$$\sup_x |\mathbb{E} F_{\mathbf{P}}(x) - \Phi(x)| \leq \frac{C\mu_3}{n^{\frac{1}{4}}},$$

and

$$\mathbb{E} \sup_x |F_{\mathbf{P}} - \Phi(x)| \leq \frac{C\mu_3}{n^{\frac{1}{4}}} + \frac{C\sqrt{\mu_4}}{n^{\frac{1}{8}}}.$$

Thus, Theorems 2 and 3 are proved.

4. ESTIMATION OF THE CONVERGENCE RATE OF THE EXPECTED SPECTRAL DISTRIBUTION FUNCTION OF A CIRCULANT AND PALINDROMIC MATRICES. PROOF OF THEOREM 4

In this section we improve the estimations of the convergence rate of the expected spectral distribution function of the matrices $\mathbf{P}$ and $\mathbf{Q}$. Moreover, we estimate the convergence rate of the distribution function of each individual eigenvalue for the circulant matrix $\mathbf{Q}$. It is known that, for a circulant matrix $\mathbf{P}$ of order $2n - 1$ with generating elements $X_1, \dots, X_n$, the eigenvalues have the form

$$\mu_j = \left(X_1 + 2 \sum_{r=2}^n \cos \left\{ \frac{2\pi(j-1)(r-1)}{n} \right\} X_r \right) / \sqrt{2n-1}. \quad (21)$$

See, for example, [5]. Obviously,

$$\begin{aligned} \mathbb{E} \mu_j^2 &= \left(1 + 4 \sum_{r=2}^n \cos^2 \left\{ \frac{2\pi(j-1)(r-1)}{n} \right\} \right) / (2n-1) \\ &= \begin{cases} 2 - \frac{2}{2n-1}, & \text{for } j = 1, \\ 1, & \text{for } j = 2, \dots, 2n-1. \end{cases} \end{aligned}$$

It is easy to see that the random variables μ_j are correlated. Indeed,

$$\sum_{j=1}^{2n-1} \mu_j = \sqrt{2n-1} X_1.$$

If the distribution of μ_j is close to normal, then the distribution of their sum can be arbitrary, coinciding with the distribution of X_1 .

Using the representation (21), the result of Theorem 2 can be refined.

Let $G_{nj}(x) = \Pr\{\mu_j \leq x\}$ be the distribution function of the random variable μ_j , $j = 1, \dots, 2n - 1$. Since μ_j represents a sum of independent random variables with finite variance, its distribution is close to normal. To estimate the proximity of the distribution $G_{nj}(x)$ to the normal distribution, one can use any version of the Berry-Esseen inequality. See, for instance, [3]. We obtain

$$\sup_x |G_{nj}(x) - \Phi(x)| \leq cL_3,$$

where

$$L_3 = \frac{1}{(2n-1)^{\frac{3}{2}}} \left(1 + 8 \sum_{r=2}^n \left| \cos \left\{ \frac{2\pi(r-1)(j-1)}{n} \right\} \right|^3 \right) \mu_3.$$

It is obvious that

$$L_3 \leq \frac{C\mu_3}{\sqrt{n}}.$$

It is easy to see that

$$\mathbb{E}F_{\mathbf{P}}(x) = \frac{1}{n} \sum_{j=1}^n G_{nj}(x).$$

From this it follows that

$$\sup_x |\mathbb{E}F_{\mathbf{P}}(x) - \Phi(x)| \leq \frac{C\mu_3}{\sqrt{n}}.$$

Thus, the inequality (12) is proved. The inequality (13) follows from the inequality (12) and (8).

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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