

Article

Limit Theorems for Spectra of Circulant Block Matrices with Large Random Blocks

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Abstract: This paper investigates the spectral properties of block circulant matrices with high-order symmetric (or Hermitian) blocks. We analyze cases with dependent or sparse independent entries within these blocks. Additionally, we analyze the distribution of singular values for the product of independent circulant matrices with non-Hermitian blocks.

Keywords: random matrices; circulant matrices; semicircle law

MSC: 60B20



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1. Circulant Block Matrices with Dependence

The study of the spectrum of circulant matrices is inspired by their diverse applications in time series analysis [1], graph theory [2–4], wireless communication [5–7], etc. The asymptotic behavior of the spectrum of circulant matrices has been studied in numerous works, such as [8–13]. In 2007, Oraby [8] (Theorem 1) investigated the limiting distribution for the empirical spectral distribution function of a circulant matrix and established the limit distribution for symmetric block circulant matrices with Wigner blocks (see [8] (Proposition 1)). In [9], the limiting spectral distribution of block symmetric circulant matrices with rectangular blocks was considered.

This paper investigates the spectral properties of block circulant matrices of three classes. First, we analyze block circulant matrices with large, symmetric (or Hermitian) blocks. We consider two cases: when the block entries exhibit dependencies and when they are independent and sparse. In the third class, we examine the distribution of singular values for the product of independent circulant matrices with non-Hermitian blocks. The proofs rely on the method of Stieltjes transform.

Let

$$C(a_1, \dots, a_k) = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-1} & a_k \\ a_k & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ a_{k-1} & a_k & a_1 & \cdots & a_{k-3} & a_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_k & a_1 \end{bmatrix}.$$

For symmetric $n \times n$ matrices $X^{(1)}, X^{(2)}, \dots, X^{(k)}$, consider circulant block matrix $W = C(\frac{1}{\sqrt{n}}X^{(1)}, \frac{1}{\sqrt{n}}X^{(2)}, \dots, \frac{1}{\sqrt{n}}X^{(k)})$. We assume that $X_j = X_{k-j+2}$ for $j = 2, \dots, k$. Under this assumption, the matrix W is symmetric and its eigenvalues are real. Let

$$h = \begin{cases} l + 1, & \text{if } k = 2l, \\ l, & \text{if } k = 2l - 1. \end{cases}$$

Note that since the matrix $\mathbf{W}$ is symmetric, the matrices $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(h)}$ determine all matrices $\mathbf{X}^{(h+1)}, \dots, \mathbf{X}^{(k)}$ with the equality $\mathbf{X}^{(h+l)} = \mathbf{X}^{(k-h-l+2)}$ for $l = 1, \dots, k-h$. Denote the eigenvalues of the matrix $\mathbf{W}$ in the following increasing order: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{nk}$. We define the empirical spectral distribution function of the matrix $\mathbf{W}$ as

$$F_n(x) = \frac{1}{kn} \sum_{j=1}^{kn} \mathbb{1}\{\lambda_j \leq x\},$$

where $\mathbb{1}\{\cdot\}$ denotes the indicator function.

We shall assume that the matrices $\mathbf{X}^{(v)}$ satisfy the following conditions for $v = 1, \dots, h$. Let $\mathbb{E} X_{ij}^{(v)} = 0$ and $\mathbb{E}(X_{ij}^{(v)})^2 = \sigma_{ij}^2$. Let $B_j^2 = \frac{1}{n} \sum_{l=1}^n \sigma_{jl}^2$. We assume that the matrices $\mathbf{X}^{(v)}$ for $v = 1, \dots, h$ are independent. Furthermore, for any $1 \leq i \leq j \leq n$ and any $v = 1, \dots, k$, introduce σ -algebras

$$\mathcal{F}_{i,j}^{(v)} = \sigma(X_{kl}^{(v)}, \text{ for } 1 \leq l < k \leq n, (k, l) \neq (i, j)).$$

We shall assume that the following conditions are satisfied:

- Condition C(1): for any $1 \leq i \leq j \leq n$ and any $v = 1, \dots, k$,

$$\mathbb{E}(X_{ij}^{(v)} | \mathcal{F}_{i,j}^{(v)}) = 0;$$

- Condition C(2): for any $v = 1, \dots, k$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E} | \mathbb{E}((X_{ij}^{(v)})^2 | \mathcal{F}_{i,j}^{(v)}) - \sigma_{ij}^2 | = 0;$$

- Condition C(3): for any $\tau > 0$ and for any $v = 1, \dots, k$,

$$\lim_{n \rightarrow \infty} L_n^{(v)}(\tau) := \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E} |X_{ij}^{(v)}|^2 \mathbb{1}\{|X_{ij}^{(v)}| \geq \tau \sqrt{n}\} = 0;$$

- Condition C(4):

$$\frac{1}{n} \sum_{j=1}^n |B_j^2 - 1| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and there exists a constant C such that $\max_{1 \leq j \leq n} B_j^2 \leq C$.

We denote the density of the semicircle law with the parameter σ^2 by $\gamma_{\sigma^2}(x)$, as follows:

$$\gamma_{\sigma^2}(x) := \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} \mathbb{1}\{|x| \leq 2\sigma\}.$$

Consider the distribution function $G^{(k)}(x)$ with the density

$$g^{(k)}(x) = \frac{d}{dx} G^{(k)}(x) = \begin{cases} \frac{k-1}{k} \gamma_{\frac{k-1}{k}}(x) + \frac{1}{k} \gamma_{\frac{2k-1}{k}}(x), & \text{if } k \text{ is odd,} \\ \frac{k-2}{k} \gamma_{\frac{k-2}{k}}(x) + \frac{2}{k} \gamma_{\frac{2k-2}{k}}(x), & \text{if } k \text{ is even.} \end{cases}$$

That is, it is a mixture of semicircle laws with parameters depending on the parity of k . For example, in the case of an odd k , we have a mixture of two semicircle laws $\gamma_{\frac{k-1}{k}}(x)$ and $\gamma_{\frac{2k-1}{k}}(x)$ with supports $[-2\sqrt{\frac{k-1}{k}}, 2\sqrt{\frac{k-1}{k}}]$ and $[-2\sqrt{\frac{2k-1}{k}}, 2\sqrt{\frac{2k-1}{k}}]$, respectively. Obviously, it is at the endpoints of these supports that the character of the function $G^{(k)}(x)$

changes. Figure 1 shows the graphs of $G^{(k)}(x)$ for $k = 3, 4, 7, 8$. The dots on the distribution function curves mark the endpoints of the supports of the original distributions.

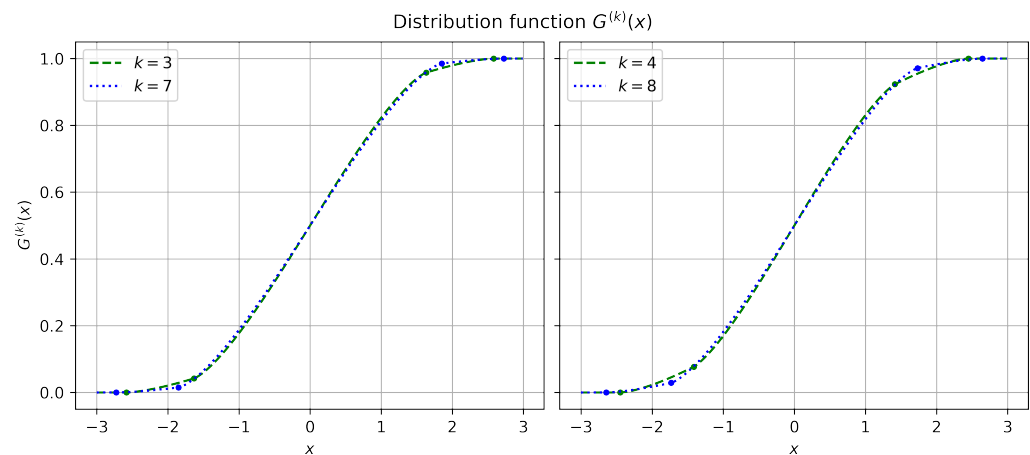


Figure 1. Distribution function $G^{(k)}(x)$ for $k = 3, 7$ (left plot) and $k = 4, 8$ (right plot).

We prove the following theorem.

Theorem 1. Under conditions C(1)–C(4), the empirical distribution function $F_n(x)$ converges in probability to the distribution function $G^{(k)}(x)$.

The proof of Theorem 1. We start from the well-known fact that the spectrum of the matrix $\mathbf{W}$ is the union of the spectra of matrices $\mathbf{B}^{(v)}$, $v = 1, \dots, k$, where

$$\mathbf{B}^{(v)} = \begin{cases} \frac{1}{\sqrt{kn}} [\mathbf{X}^{(1)} + 2 \sum_{r=2}^{\frac{k}{2}} \cos(\frac{2\pi(r-1)(v-1)}{k}) \mathbf{X}^{(r)} + \cos((v-1)\pi) \mathbf{X}^{(\frac{k}{2}+1)}], & \text{if } k \text{ is even,} \\ \frac{1}{\sqrt{kn}} [\mathbf{X}^{(1)} + 2 \sum_{r=2}^{\frac{k+1}{2}} \cos(\frac{2\pi(r-1)(v-1)}{k}) \mathbf{X}^{(r)}], & \text{if } k \text{ is odd.} \end{cases}$$

Using the orthogonal transformation, we can reduce the matrix $\mathbf{W}$ to the block-diagonal form with the blocks $\mathbf{B}^{(v)}$, $v = 1, \dots, k$, herewith the equality $\mathbf{B}^{(v)} = \mathbf{B}^{(k-v+2)}$, where $v \geq 2$, holds. Let

$$\zeta_{jl}^{(v)} = \begin{cases} \frac{1}{\sqrt{kn}} [\mathbf{X}_{jl}^{(1)} + 2 \sum_{r=2}^{\frac{k}{2}} \cos(\frac{2\pi(r-1)(v-1)}{k}) \mathbf{X}_{jl}^{(r)} + \cos((v-1)\pi) \mathbf{X}_{jl}^{(\frac{k}{2}+1)}] & \text{if } k \text{ is even,} \\ \frac{1}{\sqrt{kn}} [\mathbf{X}_{jl}^{(1)} + 2 \sum_{r=2}^{\frac{k+1}{2}} \cos(\frac{2\pi(r-1)(v-1)}{k}) \mathbf{X}_{jl}^{(r)}], & \text{if } k \text{ is odd.} \end{cases}$$

Let $\lambda_1^{(v)} \leq \lambda_2^{(v)} \leq \dots \leq \lambda_n^{(v)}$ and

$$F_n^{(v)}(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{1}\{\lambda_j^{(v)} \leq x\}.$$

It is easy to see that

$$F_n(x) = \frac{1}{k} \sum_{v=1}^k F_n^{(v)}(x).$$

We consider the case of even k because the another case is similar. Note that

$$\mathbb{E} |\zeta_{jl}^{(v)}|^2 = \frac{1}{kn} (\sigma_{jl})^2 \left[(1 + 4 \sum_{r=2}^{\frac{k}{2}} \cos^2(\frac{2\pi(r-1)(v-1)}{k}) + \cos^2((v-1)\pi)) \right].$$

For $\nu = 1$ or $\nu = k/2 + 1$, we have

$$\mathbb{E} |\zeta_{jl}^{(\nu)}|^2 = \frac{(2k-2)\sigma_{jl}^2}{kn}.$$

Using

$$\sum_{l=0}^N \cos^2(lx) = \frac{1}{2} \left(N + \frac{3}{2} + \frac{\sin(2N+1)x}{\sin x} \right),$$

we get, for $\nu \neq 1$ and $\nu \neq k/2 + 1$,

$$\mathbb{E} |\zeta_{jl}^{(\nu)}|^2 = \frac{\sigma_{jl}^2(k-2)}{kn}.$$

In fact,

$$\begin{aligned} 1 + 4 \sum_{r=2}^{\frac{k}{2}} \cos^2 \left(\frac{2\pi(r-1)(\nu-1)}{k} \right) + \cos^2((\nu-1)\pi) \\ = -2 + 4 \sum_{r=0}^{\frac{k}{2}-1} \cos^2 \left(\frac{2\pi(r-1)(\nu-1)}{k} \right) \\ = \frac{k + 2 \frac{\sin\{\frac{2\pi(k-1)(\nu-1)}{k}\}}{\sin\{\frac{2\pi(\nu-1)}{k}\}}}{k} = \frac{k-2}{k}. \end{aligned}$$

To prove Theorem 1, we only need to prove that the spectral distribution of each matrix $\mathbf{B}^{(\nu)}$, $\nu = 1, \dots, k$ converges to a semicircle law with the corresponding parameter. To this end, we apply Theorem 1 from paper [14]. We need to check the conditions (1)–(5) from ref. [14] for random variables $\zeta_{jl}^{(\nu)}$ and any fixed $\nu = 1, \dots, k$. For any fixed $\nu = 1, \dots, k$ and any $1 \leq j, l \leq n$, we introduce σ -algebras

$$\mathcal{G}_{jl}^{(\nu)} = \sigma\{\zeta_{pq} : 1 \leq p, q \leq n : (p, q) \neq (j, l)\}.$$

It is straightforward to check that, for any $\nu = 1, \dots, h$

$$\mathcal{G}_{jl}^{(\nu)} = \sigma\{\mathcal{F}_{jl}^{(\mu)}, \mu = 1, \dots, h\}.$$

Since $\mathcal{F}_{jl}^{(\mu)}$ are mutually independent for $\mu = 1, \dots, h$, we get the following using condition C(1):

$$\mathbb{E}\{\zeta_{jl}^{(\nu)} | \mathcal{G}_{jl}^{(\nu)}\} = 0.$$

Thus, condition (1) from [14] is fulfilled. Now, consider the quantity

$$Q = \frac{1}{n} \sum_{j,l=1}^n \mathbb{E} |\mathbb{E}\{(\zeta_{jl}^{(\nu)})^2 | \mathcal{G}_{jl}^{(\nu)}\} - \mathbb{E}(\zeta_{jl}^{(\nu)})^2|.$$

Assume, for instance, $1 < \nu \leq \frac{k}{2} + 1$. Note that for $j \neq p, l \neq q$,

$$\mathbb{E}\{X_{jl}^{(\mu)} X_{pq}^{(\mu)} | \mathcal{G}_{pq}\} = X_{jl}^{(\mu)} \mathbb{E}\{X_{pq} | \mathcal{G}_{pq}\} = 0.$$

Using this and the independence of $X_{jl}^{(\mu_1)}$ and $X_{pq}^{(\mu_2)}$, $\mu_1 \neq \mu_2$, we get, for an even k ,

$$\begin{aligned} \mathbb{E}\{|\zeta_{jl}^{(\nu)}|^2|\mathcal{G}_{jl}^{(\nu)}\} &= \frac{1}{kn} [\mathbb{E}\{(X_{jl}^{(1)})^2|\mathcal{G}_{jl}^{(\nu)}\} + 4 \sum_{r=2}^{\frac{k}{2}} \cos^2\left(\frac{2\pi(r-1)(\nu-1)}{k}\right) \mathbb{E}\{(X_{jl}^{(r)})^2|\mathcal{G}_{jl}^{(\nu)}\} \\ &\quad + \cos^2(2(\nu-1)\pi) \mathbb{E}\{(X_{jl}^{(\frac{k}{2}+1)})^2|\mathcal{G}_{jl}^{(\nu)}\}]. \end{aligned}$$

Applying condition C(2), we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j,l=1}^n |\mathbb{E}\{|\zeta_{jl}^{(\nu)}|^2|\mathcal{G}_{jl}^{(\nu)}\} - \mathbb{E}(\zeta_{jl}^{(\nu)})^2| = 0.$$

Thus, condition (2) from [14] is proved. Furthermore, we check condition (3) from ref. [14]. We need to prove that the Lindeberg fraction for the matrix $\mathbf{B}^{(\nu)}$ tends to zero as n tends to ∞ :

$$\lim_{n \rightarrow \infty} \tilde{L}_n^{(\nu)}(\tau) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j,l=1}^n \mathbb{E} |\zeta_{jl}^{(\nu)}|^2 \mathbb{1}\{|\zeta_{jl}^{(\nu)}| > \tau\} = 0. \quad (1)$$

We prove the following auxiliary lemma:

Lemma 1. For any random variables Y_j , $j = 1, \dots, k$ and any $b > 0$, the following inequality holds:

$$\sum_{j=1}^k Y_j^2 \mathbb{1}\left\{\sum_{j=1}^k |Y_j| > b\right\} \leq k \sum_{j=1}^k Y_j^2 \mathbb{1}\{|Y_j| > b/k\}.$$

Proof. Combining inequality $\sum_{j=1}^k |a_j| \leq k \max_{1 \leq j \leq k} |a_j|$ with inequality $\mathbb{1}\{\sum_{j=1}^k |Y_j| > b\} \leq \mathbb{1}\{\max_{1 \leq j \leq k} |Y_j| > b/k\}$, we may write

$$\sum_{j=1}^k Y_j^2 \mathbb{1}\left\{\sum_{j=1}^k |Y_j| > b\right\} \leq k \max_{1 \leq j \leq k} Y_j^2 \mathbb{1}\left\{\max_{1 \leq j \leq k} |Y_j| > b/k\right\}.$$

There exists j_0 , such that $1 \leq j_0 \leq k$ and

$$\max_{1 \leq j \leq k} Y_j^2 = Y_{j_0}^2, \text{ and } \max_{1 \leq j \leq k} |Y_j| = |Y_{j_0}|.$$

Applying this equality, we get

$$\sum_{j=1}^k Y_j^2 \mathbb{1}\left\{\sum_{j=1}^k |Y_j| > b\right\} \leq k Y_{j_0}^2 \mathbb{1}\{|Y_{j_0}| > b/k\}.$$

Applying inequality $\max_{1 \leq j \leq k} |a_j| \mathbb{1}\{|a_j| > c\} \leq \sum_{j=1}^k |a_j| \mathbb{1}\{|a_j| > c\}$, we get

$$\sum_{j=1}^k Y_j^2 \mathbb{1}\left\{\sum_{j=1}^k |Y_j| > b\right\} \leq k \sum_{j=1}^k Y_j^2 \mathbb{1}\{|Y_j| > b/k\}.$$

Thus, Lemma 1 is proved. $\square$

Now, we continue with the proof of relation (1). It is straightforward to check that

$$|\zeta_{jl}^{(\nu)}| \leq \frac{2}{\sqrt{kn}} \sum_{r=1}^{\frac{k}{2}+1} |X_{jl}^{(r)}|, \text{ and } |\zeta_{jl}^{(\nu)}|^2 \leq \frac{2}{n} \sum_{r=1}^{\frac{k}{2}+1} |X_{jl}^{(r)}|^2.$$

This implies that

$$\mathbb{E} |\zeta_{jl}^{(v)}|^2 \mathbb{1}\{|\zeta_{jl}| > \tau\} \leq \frac{2}{n} \mathbb{E} \left(\sum_{r=1}^{\frac{k}{2}+1} |X_{jl}^{(r)}|^2 \right) \mathbb{1}\left\{ \sum_{r=1}^{\frac{k}{2}+1} |X_{jl}^{(r)}| > \tau\sqrt{kn} \right\}.$$

Now, by applying Lemma 1, we get

$$\mathbb{E} |\zeta_{jl}^{(v)}|^2 \mathbb{1}\{|\zeta_{jl}| > \tau\sqrt{n}\} \leq \frac{k}{n} \sum_{k=1}^h \mathbb{E} |X_{jl}^{(r)}|^2 \mathbb{1}\{|X_{jl}^{(r)}| > \tau\sqrt{n/k}\}.$$

Hence,

$$\tilde{L}_n^{(v)}(\tau) \leq k \sum_{r=1}^k L_n^{(r)}(\tau/\sqrt{k}).$$

This inequality and condition C(3) proved condition (3) from ref. [14].

Let $(\tilde{B}_j^{(v)})^2 := \sum_{l=1}^n \mathbb{E} \zeta_{jl}^{(v)2}$. We need to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n |(\tilde{B}_j^{(v)})^2 - d_v| = 0, \quad (2)$$

where

$$d_v = \begin{cases} \frac{k-2}{k}, & \text{if } v = 1 \text{ or } k/2 + 1, \\ \frac{2k-2}{k}, & \text{otherwise.} \end{cases}$$

It is straightforward to see that

$$(\tilde{B}_j^{(v)})^2 = \frac{d_v}{n} \sum_{l=1}^n \sigma_{jl}^2 = d_v B_j^2.$$

Relation (2) and condition (5) from [14] now follow from condition C(4). Thus, conditions (1)–(5) from ref. [14] are fulfilled, and the result of Theorem 1 now follows from ref. [14] (Theorem 1.1). $\square$

2. Circulant Block Matrices with Sparsity

In this section, we consider circulant block matrices with independent blocks $\mathbf{X}^{(v)}$, $v = 1, \dots, h$. We shall assume as well that the entries of matrices $\mathbf{X}^{(v)}$ are independent and sparse with independent Bernoulli random variables $\xi_{jl}^{(v)}$, $\mathbb{E} \xi_{jl}^{(v)} = p_{jl}^{(v)}$. We construct the blocks

$$\tilde{\mathbf{W}}^{(v)} = (\mathbf{W}_{jl}^{(v)}) := \mathbf{X}^{(v)} \circ \Xi^{(v)}, \quad \mathbf{W}_{jl}^{(v)} = X_{jl}^{(v)} \xi_{jl}^{(v)},$$

where “ $\circ$ ” denotes the Hadamard product of matrices and $\Xi = (\xi_{jl}^{(v)})_{j,l=1}^n$. Recall that all random variables $X_{jl}^{(v)}$, $\xi_{jl}^{(v)}$, $j, l = 1, \dots, n$, $v = 1, \dots, h$ are jointly independent. For simplicity, we shall assume that $\mathbb{E}(X_{jl}^{(v)})^2 = (\sigma_{jl}^{(v)})^2 = \sigma_{jl}^2$ and $p_{jl}^{(v)} = p_{jl}$ for any $v = 1, \dots, h$. We introduce the normalizing factor

$$a_n := \frac{1}{n} \sum_{j,l=1}^n p_{jl} \sigma_{jl}^2,$$

and, for the blocks,

$$\mathbf{W}^{(v)} = \frac{1}{\sqrt{a_n}} \tilde{\mathbf{W}}^{(v)}$$

we define the block matrix

$$\mathbf{Y} := \mathbf{C}(\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(h)}),$$

where $\mathbf{W}^{(h+\nu)} = \mathbf{W}^{(k-h-\nu+2)}$. For instance, $\mathbf{W}^{(h+1)} = \mathbf{W}^{(h-1)}, \dots, \mathbf{W}^{(k)} = \mathbf{W}^{(2)}$. We introduce some conditions on the probabilities p_{jl} and on the distributions of $X_{jl}^{(\nu)}$ for $\nu = 1, \dots, h$ and $j, l = 1, \dots, n$. Recall that

$$p_{jl} = \mathbb{E} \xi_{jl}^{(\nu)}, \text{ and } \sigma_{jl}^2 = \mathbb{E} (X_{jl}^{(\nu)})^2.$$

We shall assume that the following conditions are satisfied:

- Condition D(1):

$$a_n \rightarrow \infty \text{ as } n \rightarrow \infty;$$

- Condition D(2):

$$\frac{1}{n} \sum_{j=1}^n \sum_{l=1}^n |p_{jl} \sigma_{jl} - \frac{a_n}{n}| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } \nu = 1, \dots, k;$$

- Condition D(3):

$$\lim_{n \rightarrow \infty} L_n(\tau) := \lim_{n \rightarrow \infty} \frac{1}{na_n} \sum_{\nu=1}^k \sum_{j=1}^n \sum_{l=1}^n p_{jl} \mathbb{E} (X_{jl}^{(\nu)})^2 \mathbb{1}_{\{|X_{jl}^{(\nu)}| \geq \tau \sqrt{a_n}\}} = 0;$$

- Condition D(4):

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq n} p_{jj} \sigma_{jj}^2 / a_n = 0,$$

and there exists a constant C_0 such that for any $n \geq 1$

$$\max_{1 \leq i, j \leq n} p_{ij} \sigma_{ij}^2 / a_n \leq C_0.$$

We prove the following theorem.

Theorem 2. Let conditions D(1)–D(4) be fulfilled. Then,

$$\lim_{n \rightarrow \infty} F_n(x) = G^{(k)}(x),$$

where $G^{(k)}(x)$ was defined in Theorem 1.

The proof of Theorem 2. We recall that the matrix $\mathbf{W}$ can be reduced by a unitary (orthogonal) transformation to the block-diagonal form with the blocks $\mathbf{B}^{(\nu)}$, $\nu = 1, \dots, k$, such that $\mathbf{B}^{(\nu)} = \mathbf{B}^{(k-\nu+2)}$ and, for $\nu = 2, \dots, k$,

$$\mathbf{B}^{(\nu)} = \frac{1}{\sqrt{k}} \left[\mathbf{W}^{(1)} + 2 \sum_{r=2}^{\frac{k+1}{2}} \cos \left(\frac{2\pi(r-1)(\nu-1)}{k} \right) \mathbf{W}^{(r)} \right],$$

if k is odd, and

$$\mathbf{B}^{(\nu)} = \frac{1}{\sqrt{k}} \left[\mathbf{W}^{(1)} + 2 \sum_{r=2}^{\frac{k}{2}} \cos \left(\frac{2\pi(r-1)(\nu-1)}{k} \right) \mathbf{W}^{(r)} + \cos((\nu-1)\pi) \mathbf{W}^{(\frac{k}{2}+1)} \right],$$

if k is even. To prove the theorem, it is enough to prove that the spectral distribution functions of the matrices $\mathbf{B}^{(\nu)}$ converges to the semicircle law with the corresponding parameter. Introduce the resolvent matrix of matrix $\mathbf{B}^{(\nu)}$,

$$\mathbf{R}^{(\nu)}(z) = (\mathbf{W}^{(\nu)} - z\mathbf{I})^{-1}.$$

Denote the Stieltjes transform of the spectral distribution of the matrix $\mathbf{W}^{(v)}$ by $s_n^{(v)}(z)$. We have

$$s_n^{(v)}(z) = \frac{1}{n} \sum_{j=1}^n R_{jj}^{(v)}.$$

Applying Schur's complement formula, we get

$$R_{jj}^{(v)}(z) = \frac{1}{\frac{1}{\sqrt{a_n}} B_{jj}^{(v)} - z - \frac{1}{a_n} \sum_{l=1, l \neq j}^n \sum_{q=1, q \neq j}^n B_{jl}^{(v)} B_{jq}^{(v)} R_{lq}^{(v,j)}}, \quad (3)$$

where $\mathbf{B}^{(v,j)}$ denotes the matrix obtained from $\mathbf{B}^{(v)}$ by deleting the j -th row and column, and

$$\mathbf{R}^{(v,j)} = \mathbf{R}^{(v,j)}(z) = (\mathbf{B}^{(v,j)} - z\mathbf{I})^{-1}.$$

Introduce the following notations:

$$\begin{aligned} \varepsilon_{j1}^{(v)} &= \frac{1}{\sqrt{a_n}} B_{jj}^{(v)}, \\ \varepsilon_{j2}^{(v)} &= \frac{1}{a_n} \sum_{l=1, l \neq j}^n ((B_{jl}^{(v)})^2 - \mathbb{E}(B_{jl}^{(v)})^2) R_{ll}^{(v,j)}, \\ \varepsilon_{j3}^{(v)} &= \frac{1}{a_n} \sum_{1 \leq l, q \leq n, l \neq j, q \neq j, p \neq q} B_{jl}^{(v)} B_{jq}^{(v)} R_{lq}^{(v,j)}. \end{aligned}$$

It is straightforward to check that, for $j, l = 1, \dots, n$ and $v = 1, \dots, h$, when k is odd,

$$\mathbb{E}(B_{jl}^{(v)})^2 = \sigma_{jl}^2 p_{jl} \left(2 + 2 \sum_{r=2}^h \cos^2 \left(\frac{2\pi(r-1)(v-1)}{k} \right) \right) = \sigma_{jl}^2 p_{jl} d_v a_n^{-1}. \quad (4)$$

Let

$$A_{1j}(z) = \frac{d_v}{a_n} \sum_{l=1, l \neq j}^n \sigma_{jl}^2 p_{jl} R_{ll}^{(v,j)}.$$

Now, we can rewrite equality (3) in the form

$$R_{jj}^{(v)}(z) = \frac{1}{-z - A_{1j}(z) + \varepsilon_{j1}^{(v)} + \varepsilon_{j2}^{(v)} + \varepsilon_{j3}^{(v)}}.$$

Now, we approximate $A_{1j}(z)$ using condition $D(2)$. Introduce

$$\begin{aligned} \varepsilon_{j4}^{(v)} &= \frac{d_v}{a_n} \sum_{l=1, l \neq j}^n \left(\sigma_{jl}^2 p_{jl} - \frac{a_n}{n} \right) R_{ll}^{(v,j)}, \\ \varepsilon_{j5}^{(v)} &= d_v \left(\frac{1}{n} \sum_{l=1, l \neq j}^n R_{ll}^{(v,j)} - \frac{1}{n} \sum_{l=1}^n R_{ll}^{(v)} \right). \end{aligned}$$

Using $s_n^{(v)}(z) = \frac{1}{n} \sum_{l=1}^n R_{ll}^{(v)}$ and

$$\varepsilon_j^{(v)} = \varepsilon_{j1}^{(v)} + \dots + \varepsilon_{j5}^{(v)},$$

we may write

$$R_{jj}^{(v)}(z) = \frac{1}{-z - d_v s_n^{(v)}(z) + \varepsilon_j^{(v)}}. \quad (5)$$

Summing the last equality in $j = 1, \dots, n$, we get

$$s_n^{(v)}(z) = -\frac{1}{z + d_v s_n^{(v)}(z)} + T_n^{(v)}(z),$$

where

$$T_n^{(v)}(z) = \frac{1}{z + d_v s_n^{(v)}(z)} \frac{1}{n} \sum_{j=1}^n \varepsilon_j^{(v)} R_{jj}^{(v)}(z).$$

To prove the convergence

$$\lim_{n \rightarrow \infty} s_n^{(v)}(z) = s^{(v)}(z) \text{ in probability,}$$

it is enough to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E} |T_n(z)| = 0.$$

Here, $s^{(v)}(z)$ denotes the Stieltjes transform of the semicircle law with the parameter $d^{(v)}$.

The inequality $|R_{jj}^{(v)}(z)| \leq \frac{1}{v}$ for $z = u + iv$ implies that

$$|T_n^{(v)}(z)| \leq \frac{1}{v^2} \frac{1}{n} \sum_{j=1}^n |\varepsilon_j^{(v)}|.$$

The last inequality implies that it is enough to prove

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_j^{(v)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We start with $\varepsilon_{j1}^{(v)}$.

Lemma 2. Under the conditions of Theorem 1, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j1}^{(v)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Since the cases of an odd and of even k are similar, we only consider an odd k . By the definition of ε_{j1} and $\mathbf{B}^{(v)}$, for odd k we have

$$\varepsilon_{j1}^{(v)} = \frac{1}{k\sqrt{a_n}} \left(X_{jj}^{(1)} \zeta_{jj} + 2 \sum_{r=2}^h \cos \left(\frac{2\pi(v-1)(r-1)}{k} \right) X_{jj}^{(r)} \zeta_{jj}^{(r)} + \cos(2\pi(v-1)) X_{jj}^{(h+1)} \zeta_{jj}^{(h+1)} \right).$$

Applying Cauchy's inequality, we get

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j1}^{(v)}| \leq \left(\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j1}^{(v)}|^2 \right)^{\frac{1}{2}}.$$

Furthermore,

$$\mathbb{E} |\varepsilon_{j1}^{(v)}|^2 \leq \frac{1}{ka_n} \sigma_{jj}^2 p_{jj} \left(2 + 4 \sum_{r=2}^h \cos^2 \left(\frac{2\pi(v-1)(r-1)}{k} \right) \right) \leq \frac{2}{a_n} p_{jj} \sigma_{jj}^2.$$

Summing up this inequality by indices $j = 1, \dots, n$, we get

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j1}^{(v)}| \leq \left(\frac{2}{na_n} \sum_{j=1}^n p_{jj} \sigma_{jj}^2 \right)^{\frac{1}{2}}.$$

Continuing with this inequality, we get

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j1}^{(\nu)}| \leq \left(\frac{2 \max_{1 \leq j \leq n} p_{jj} \sigma_{jj}^2}{a_n} \right)^{\frac{1}{2}}.$$

The last inequality and condition D(4) imply the result. Thus, Lemma 2 is proved. $\square$

Lemma 3. Under the conditions of Theorem 2, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j2}^{(\nu)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. We start with the definition of $\varepsilon_{j2}^{(\nu)}$. Thus, we have

$$\varepsilon_{j2}^{(\nu)} = \frac{1}{a_n} \sum_{l=1, l \neq j}^n ((B_{jl}^{(\nu)})^2 - \mathbb{E}(B_{jl}^{(\nu)})^2) R_{ll}^{(\nu, j)}.$$

Note that according to (4),

$$\mathbb{E}(B_{jl}^{(\nu)})^2 = \mathbb{E}(B_{jl}^{(\nu)})^2 = \sigma_{jl}^2 p_{jl} \left(2 + 2 \sum_{r=2}^h \cos^2 \left(\frac{2\pi(r-1)(\nu-1)}{k} \right) \right) = \sigma_{jl}^2 p_{jl} d_\nu. \quad (6)$$

We may rewrite $\varepsilon_{j2}^{(\nu)}$ as follows:

$$\varepsilon_{j2}^{(\nu)} = f_{j1}^{(\nu)} + f_{j2}^{(\nu)} + f_{j3}^{(\nu)},$$

where

$$\begin{aligned} f_{j1}^{(\nu)} &= \frac{1}{k} \frac{1}{a_n} \sum_{l=1}^n ((X_{jl}^{(1)})^2 \xi_{jl}^{(1)} - p_{jl} \sigma_{jl}^2), \\ f_{j2}^{(\nu)} &= \frac{1}{k} \sum_{r=2}^k \frac{1}{a_n} \sum_{l=1}^n \cos^2 \left(\frac{2\pi(\nu-1)(r-1)}{k} \right) (X_{jl}^{(r)})^2 \xi_{jl}^{(r)} - p_{jl} \sigma_{jl}^2, \\ f_{j3}^{(\nu)} &= \frac{4}{k} \frac{1}{a_n} \sum_{l=1}^n \sum_{r, q=1, r \neq q}^h \cos \left(\frac{2\pi(\nu-1)(r-1)}{k} \right) \cos \left(\frac{2\pi(\nu-1)(q-1)}{k} \right) X_{jl}^{(r)} \xi_{jl}^{(r)} X_{jl}^{(q)} \xi_{jl}^{(q)}. \end{aligned}$$

To estimate $f_{j1}^{(\nu)}$, we represent it in the following form:

$$f_{j1}^{(\nu)} = \bar{f}_{j1}^{(\nu)} + \tilde{f}_{j1}^{(\nu)},$$

where

$$\begin{aligned} \bar{f}_{j1}^{(\nu)} &= \frac{1}{k} \frac{1}{a_n} \sum_{l=1}^n ((X_{jl}^{(1)})^2 \xi_{jl}^{(1)} - p_{jl} \sigma_{jl}^2) \mathbb{1}_{\{|X_{jl}^{(1)}| > \tau \sqrt{a_n}\}}, \\ \tilde{f}_{j1}^{(\nu)} &= \frac{1}{k} \frac{1}{a_n} \sum_{l=1}^n ((X_{jl}^{(1)})^2 \xi_{jl}^{(1)} - p_{jl} \sigma_{jl}^2) \mathbb{1}_{\{|X_{jl}^{(1)}| \leq \tau \sqrt{a_n}\}}. \end{aligned}$$

For $\bar{f}_{j1}^{(v)}$, we have the estimation

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbb{E} |\bar{f}_{j1}^{(v)}| &\leq \frac{1}{nka_n} \sum_{j=1}^n \sum_{l=1}^n \mathbb{E} |(X_{jl}^{(1)})^2 \zeta_{jl}^{(1)} - p_{jl} \sigma_{jl}^2| \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} \\ &\leq \frac{1}{nka_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl} \mathbb{E} (X_{jl}^{(1)})^2 \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} \\ &\quad + \frac{1}{nka_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl}^2 \sigma_{jl}^2 \mathbb{E} \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} \\ &\leq \frac{1}{k} L_n(\tau) + \frac{1}{nka_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl}^2 \sigma_{jl}^2 \mathbb{E} \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\}. \end{aligned} \quad (7)$$

Note that

$$\sigma_{jl}^2 \leq \tau^2 a_n + \mathbb{E} |X_{jl}^{(1)}|^2 \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\}.$$

Using this inequality, we get

$$\begin{aligned} \frac{1}{nka_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl}^2 \sigma_{jl}^2 \mathbb{E} \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} &\leq \frac{2}{nka_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl} \mathbb{E} (X_{jl}^{(1)})^2 \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} \\ &= \frac{2}{k} L_n(\tau). \end{aligned} \quad (8)$$

Here, we use inequalities $p_{jl} \leq 1$, $a_n \tau^2 \mathbb{E} \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\} \leq \mathbb{E} (X_{jl}^{(1)})^2 \mathbb{1}\{|X_{jl}^{(1)}| > \tau\sqrt{a_n}\}$, and $\mathbb{1}\{|X_{jl}| > \tau\sqrt{a_n}\} \leq 1$. Combining inequalities (7) and (8), we obtain

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\bar{f}_{j1}^{(v)}| \leq \frac{3}{k} L_n(\tau).$$

Similarly, get the bound

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\bar{f}_{j2}^{(v)}| \leq 3L_n(\tau).$$

Furthermore,

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |f_{j3}^{(v)}| \leq \left(\frac{1}{n} \sum_{j=1}^n \mathbb{E} |f_{j3}^{(v)}|^2 \right)^{\frac{1}{2}}.$$

Using the independence of summands on the right hand side in reference with $f_{j3}^{(v)}$ and that $|\cos \phi| \leq 1$, we get

$$\mathbb{E} |f_{j3}^{(v)}|^2 \leq \frac{k}{a_n^2} \sum_{l=1}^n p_{jl}^2 \sigma_{jl}^4.$$

Applying this inequality, we get

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |f_{j3}^{(v)}|^2 \leq \frac{1}{na_n^2} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^4 p_{jl}^2.$$

The inequality

$$\sigma_{jl}^2 \leq \tau^2 a_n + \mathbb{E} |X_{jl}^{(v)}|^2 \mathbb{1}\{|X_{jl}^{(v)}|^2 > \tau\sqrt{a_n}\} \quad (9)$$

implies that

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |f_{j3}^{(v)}|^2 \leq \frac{\tau^2}{na_n} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl}^2 + \frac{1}{na_n^2} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl}^2 \mathbb{E} |X_{jl}^{(v)}|^2 \mathbb{1}\{|X_{jl}^{(v)}|^2 > \tau\sqrt{a_n}\}.$$

For the first term in the r.h.s. of this inequality, we have

$$\frac{\tau^2}{na_n} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl}^2 \leq \frac{\tau^2}{na_n} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl} \leq \tau^2.$$

Using inequality (9) again, we get

$$\frac{1}{na_n^2} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl}^2 \mathbb{E} |X_{jl}^{(v)}|^2 \mathbb{1}\{|X_{jl}^{(v)}|^2 > \tau \sqrt{a_n}\} \leq \frac{C_0}{na_n} \sum_{j=1}^n \sum_{l=1}^n p_{jl} \mathbb{E} |X_{jl}^{(v)}|^2 \mathbb{1}\{|X_{jl}^{(v)}| > \tau \sqrt{a_n}\}.$$

Using the definition of $L_n(\tau)$, we have

$$\frac{1}{na_n^2} \sum_{j=1}^n \sum_{l=1}^n \sigma_{jl}^2 p_{jl}^2 \mathbb{E} |X_{jl}^{(v)}|^2 \mathbb{1}\{|X_{jl}^{(v)}|^2 > \tau \sqrt{a_n}\} \leq C_0 L_n(\tau).$$

The claim follows from condition $D(3)$. $\square$

Lemma 4. Under the conditions of Theorem 2, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j3}^{(v)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Applying Cauchy's inequality, we get

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j3}^{(v)}| \leq \left(\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j3}^{(v)}|^2 \right)^{\frac{1}{2}}.$$

Furthermore, the independence of random variables $B_{jl}^{(v)}$, $B_{jq}^{(v)}$, and $R_{lq}^{(v,j)}$ for $l \neq q$ implies that

$$\mathbb{E} |\varepsilon_{j3}^{(v)}|^2 \leq \frac{1}{a_n^2} \sum_{l=1}^n \sum_{q=1}^n \mathbb{E} (B_{jl}^{(v)})^2 \mathbb{E} (B_{jq}^{(v)})^2 \mathbb{E} |R_{lq}^{(v,j)}|^2.$$

Applying equality (6), we obtain

$$\mathbb{E} |\varepsilon_{j3}^{(v)}|^2 \leq \frac{(d^{(v)})^2}{a_n^2} \sum_{l=1}^n \sum_{q=1}^n \sigma_{jl}^2 \sigma_{jq}^2 p_{jl} p_{jq} \mathbb{E} |R_{lq}^{(v,j)}|^2.$$

$\square$

Lemma 5. Under the conditions of Theorem 2, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j4}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Using the definition of ε_{j4} and the inequality $|R_{jj}| \leq v^{-1}$, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j4}| \leq \frac{d_v}{vna_n} \sum_{j=1}^n \sum_{l=1, l \neq j}^n \left| \sigma_{jl}^2 p_{jl} - \frac{a_n}{n} \right|.$$

Condition $D(2)$ completes the proof. $\square$

Lemma 6. Under the conditions of Theorem 2, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j5}^{(v)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Using the definition of $\varepsilon_{j5}^{(v)}$, we have

$$\frac{1}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{j5}^{(v)}| \leq d_v \left(\frac{1}{n} \sum_{j=1}^n \left| \frac{1}{n} \sum_{l=1, l \neq j}^n R_{ll}^{(v,j)} - \frac{1}{n} \sum_{l=1}^n R_{ll}^{(v)} \right| \right).$$

It is well known (see, for instance, ref. [15]) (Lemma 5.2) that

$$\left| \frac{1}{n} \sum_{l=1, l \neq j}^n R_{ll}^{(v,j)} - \frac{1}{n} \sum_{l=1}^n R_{ll}^{(v)} \right| \leq \frac{1}{nv}.$$

This inequality implies the result of Lemma 6. Thus, the lemma is proved. $\square$

Combining the results of Lemmas 2–6, we get

$$\frac{1}{n} \sum_{j=1}^n |\varepsilon_j^{(v)}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, Theorem 2 is proved. $\square$

3. Product of Circulant Matrices

In this section, we consider the circulant matrices without symmetry. Let

$$\mathbf{C}(a_1, \dots, a_k) = \frac{1}{\sqrt{k}} \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-1} & a_k \\ a_k & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ a_{k-1} & a_k & a_1 & \cdots & a_{k-3} & a_{k-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \cdots & a_k & a_1 \end{bmatrix}.$$

Consider the array of random $n \times n$ matrices $\{\mathbf{X}^{v,\mu} = (X_{jl}^{(v,\mu)}), v = 1, \dots, k, \mu = 1, \dots, m\}$. We shall assume that all random variables are independent in aggregate. With regard to random variables $X_{jl}^{(v,\mu)}$, we shall assume that the following conditions are satisfied:

- Condition E(1): For any $n \geq 1$ and $j, l = 1, \dots, n, v = 1, \dots, k, \mu = 1, \dots, m$,

$$\mathbb{E} X_{jl}^{(v,\mu)} = 0, \quad \mathbb{E} |X_{jl}^{(v,\mu)}|^2 = 1;$$

- Condition E(2): For any $v = 1, \dots, k, \mu = 1, \dots, m$ and for any $\tau > 0$, we have

$$L_n(\tau) = \frac{1}{n^2} \sum_{j=1}^n \sum_{l=1}^n \mathbb{E} |X_{jl}^{(v,\mu)}|^2 \mathbb{1}\{|X_{jl}^{(v,\mu)}| > \tau \sqrt{n}\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let us introduce the matrices

$$\mathbf{Y}^{(\mu)} = \frac{1}{\sqrt{n}} \mathbf{C}(\mathbf{X}^{(1,\mu)}, \mathbf{X}^{(2,\mu)}, \dots, \mathbf{X}^{(k,\mu)}).$$

We shall investigate the distribution of singular values of the matrix

$$\mathbf{W} = \prod_{\mu=1}^m \mathbf{Y}^{(\mu)}.$$

Denote the singular values of the matrix $\mathbf{W}$ by $s_1^2 \geq s_2^2 \geq \dots \geq s_{kn}^2$ and the spectral distribution function of the matrix $\mathbf{W}\mathbf{W}^*$ by $G_n^{(m)}(x)$, as follows:

$$G_n^{(m)}(x) = \sum_{j=1}^{kn} \mathbb{1}\{s_j^2 \leq x\}.$$

We prove the following result.

Theorem 3. Under conditions E(1)–E(2), we have

$$\lim_{n \rightarrow \infty} G_n^{(m)}(x) = G^{(m)}(x),$$

where $G^{(m)}(x)$ is the distribution function described via its Stieltjes transform $s^{(m)}(z)$ that satisfies the equation

$$1 + zs^{(m)}(z) + (-1)^{m+1}(s^{(m)}(z))^m = 0.$$

Proof. First of all, we note that the matrices $\mathbf{Y}^{(\mu)}$ can be transformed into a block diagonal form by applying the block matrix $\mathbf{U}$ defined as

$$\mathbf{U} = \begin{bmatrix} \omega_1^0 \mathbf{I}_n & \omega_2^0 \mathbf{I}_n & \omega_3^0 \mathbf{I}_n & \dots & \omega_{k-1}^0 \mathbf{I}_n \\ \omega_1^1 \mathbf{I}_n & \omega_2^1 \mathbf{I}_n & \omega_3^1 \mathbf{I}_n & \dots & \omega_{k-1}^1 \mathbf{I}_n \\ \dots & \dots & \dots & \dots & \dots \\ \omega_1^{k-1} \mathbf{I}_n & \omega_2^{k-1} \mathbf{I}_n & \omega_3^{k-1} \mathbf{I}_n & \dots & \omega_{k-1}^{k-1} \mathbf{I}_n \end{bmatrix},$$

where $\omega_j = e^{\frac{i2\pi j}{k}}$ and $\mathbf{I}_n$ denotes the unit matrix of order n . Consequently, $\mathbf{W}$ becomes a block-circulant matrix that can also be reduced to a block-diagonal form with $\mathbf{U}$:

$$\text{diag}\{\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \dots, \mathbf{B}^{(k)}\} = \mathbf{U}\mathbf{W}\mathbf{U}^*,$$

where $\text{diag}\{\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \dots, \mathbf{B}^{(k)}\}$ denotes a diagonal-block matrix with blocks $\mathbf{B}^{(v)}$ of order $n \times n$. These blocks are the product of matrices $\mathbf{B}^{(v,\mu)}$:

$$\mathbf{B}^{(v)} = \prod_{\mu=1}^m \mathbf{B}^{(v,\mu)},$$

with

$$\mathbf{B}^{(v,\mu)} = \frac{1}{\sqrt{nk}} \sum_{r=1}^k \omega_v^{r-1} \mathbf{X}^{(r,\mu)}.$$

Introduce matrix

$$\mathbf{V} = \text{diag}\{\mathbf{B}^{(1)}\mathbf{B}^{(1)*}, \mathbf{B}^{(2)}\mathbf{B}^{(2)*}, \dots, \mathbf{B}^{(k)}\mathbf{B}^{(k)*}\}.$$

The spectrum of matrix $\mathbf{V}$ coincides with the spectrum of $\mathbf{W}\mathbf{W}^*$. Furthermore, it is the union of the spectra of matrices $\mathbf{B}^{(v)}\mathbf{B}^{(v)*}$ for $v = 1, \dots, k$.

Let $s_l^{(v)^2}$ denote the eigenvalues of $\mathbf{B}^{(v)}\mathbf{B}^{(v)*}$ for $l = 1, \dots, n$, $v = 1, \dots, k$. We define the empirical spectral distribution function $G_{nv}(x)$ as

$$G_{nv}(x) = \frac{1}{n} \sum_{l=1}^n \mathbb{1}\{s_l^{(v)^2} \leq x\}.$$

It is simple to see that

$$G_n^{(m)}(x) = \frac{1}{k} \sum_{v=1}^k G_{nv}(x). \quad (10)$$

We define $s_n^{(m)}(z)$ and $s_n^{(v)}(z)$ as the Stieltjes transforms of the distribution functions $G_n^{(m)}(x)$ and $G_n^{(v)}(x)$, respectively. Using equality (10), we have

$$s_n^{(m)}(z) = \frac{1}{k} \sum_{v=1}^k s_n^{(v)}(z). \quad (11)$$

Consider the entries of $\mathbf{B}^{(v,\mu)} = (B_{jl}^{(v,\mu)})_{j,l=1}^n$ for $\mu = 1, \dots, m$. It is straightforward to check that

$$\mathbb{E} |B_{jl}^{(v,\mu)}|^2 = \frac{1}{n}.$$

Since for each fixed value of v , the random matrices $\mathbf{B}^{(v,1)}, \mathbf{B}^{(v,2)}, \dots, \mathbf{B}^{(v,m)}$ are jointly independent and the entries of each individual matrix are also independent, we may apply (Theorem 1.1) [16] the case of square matrices, where $y_1 = y_2 = \dots = y_m = 1$. We need to check the Lindeberg condition for random variables $B_{jl}^{(v,\mu)}$, for $j, l = 1, \dots, n$ and $\mu = 1, \dots, m$. To verify this condition, we can apply Lemma 1 in a similar manner to the symmetric case. Note that

$$\begin{aligned} \mathbb{E} |B_{jl}^{(v,\mu)}|^2 \mathbb{1}\{|B_{jl}^{(v,\mu)}| > \tau\sqrt{n}\} &\leq \sum_{r=1}^k \mathbb{E} |X^{(r,\mu)}|^2 \mathbb{1}\left\{\sum_{l=1}^k |X^{(l,\mu)}| > \tau\sqrt{n}\right\} \\ &\leq k \sum_{r=1}^k \mathbb{E} |X^{(r,\mu)}|^2 \mathbb{1}\{|X^{(r,\mu)}| > \tau\sqrt{n}/k\}. \end{aligned}$$

The last inequality and condition $E(1)$ together imply that

$$\begin{aligned} \frac{1}{n^2} \sum_{j,l=1}^n \mathbb{E} |B_{jl}^{(v,\mu)}|^2 \mathbb{1}\{|B_{jl}^{(v,\mu)}| > \tau\sqrt{n}\} &\leq \sum_{r=1}^k \frac{1}{n^2} \sum_{j,l=1}^n \mathbb{E} |X^{(r,\mu)}|^2 \mathbb{1}\{|X^{(r,\mu)}| > \tau\sqrt{n}/k\} \\ &= \sum_{r=1}^k L_n^{(r,\mu)}(\tau/k) \rightarrow 0 \text{ as } \rightarrow \infty. \end{aligned}$$

From (Theorem 1) [16], it follows that

$$s_n^{(v)}(z) \rightarrow s^{(m)}(z), \text{ for any } z \in \mathbb{C}_+.$$

Equality (11) now implies the result of the theorem. Thus, Theorem 3 is proved. $\square$

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